QMDD and Spectral Transformation of Binary and Multiple-Valued Functions

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Outline

- Spectral Transformations
- Spectral Transformation Computation
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- QMDD-based Spectral Transformation
- Experimental Results
- Concluding Remarks

Spectral Transformations

$$S = T^n F$$

$$T^n = \bigotimes_{i=1}^{n} T^1$$

Example Transformations

- **Rademacher-Walsh**: $T^1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
- **Reed-Muller**: $T^1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$
- **Chrestenson**: $T^1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix}, \; a = e^{\frac{-2\pi i}{r}}, \; r = 3$

Spectral Transformation Computation

- Matrix multiplication $O(r^{2n})$
- ‘Fast’ transformation techniques

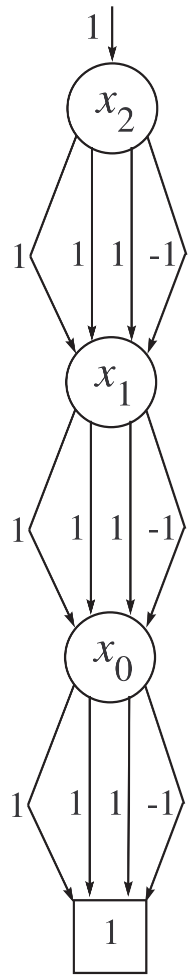
```
f[k]=(f[j]+f[k])&1
```

```
void fht(int f[], int n)
{
  int i,j,k,t,m,p;
  for (m=1;m<(1<<n);m=m<<1)
  {
    for (i=0;i<(1<<n);i+=m<<1)
    {
      for (j=i,p=k=i+m;j<p;j++,k++)
      {
        t=f[j];
        f[j]=(f[j]+f[k]);
        f[k]=(t-f[k]);
      }
    }
  }
}
```

QMDD

$$M = \begin{bmatrix} M_0 & M_1 & \cdots & M_{r-1} \\ M_r & M_{r+1} & \cdots & M_{2r-1} \\ \vdots & \vdots & \ddots & \vdots \\ M_{r^2-r} & M_{r^2-r+1} & \cdots & M_{r^2-1} \end{bmatrix}$$

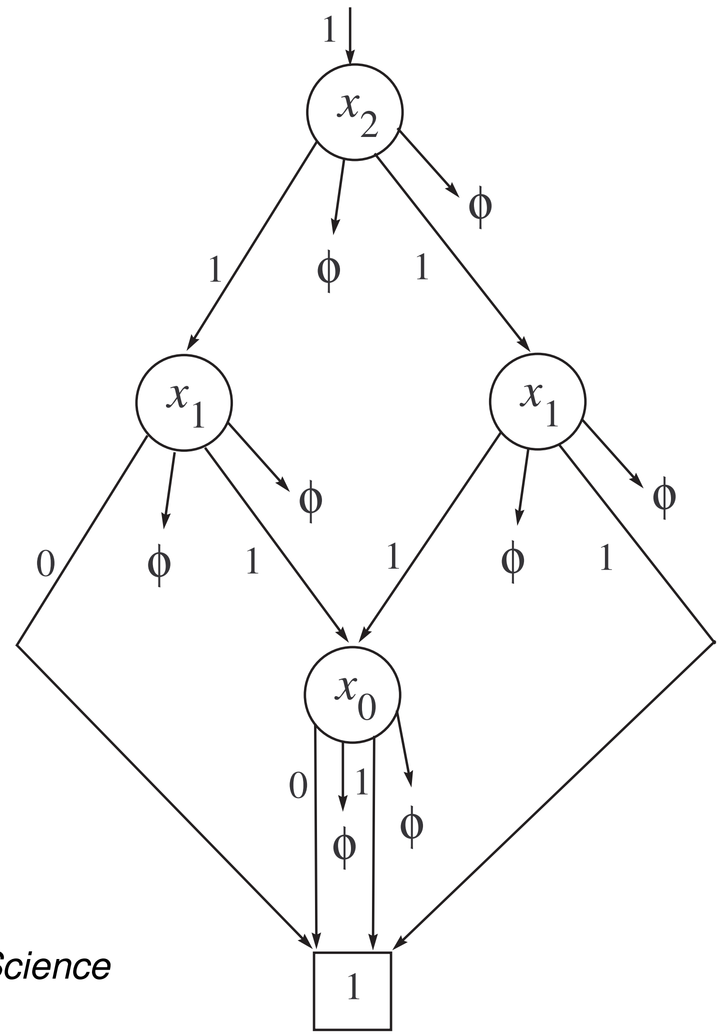
QMDD for Rademacher-Walsh T^3



QMDD for Column Vectors

$$V = \begin{bmatrix} V_0 & \phi & \Lambda & \phi \\ V_r & \phi & \Lambda & \phi \\ \mathrm{M} & \mathrm{M} & \mathrm{O} & \mathrm{M} \\ V_{r^2-r} & \phi & \Lambda & \phi \end{bmatrix}$$

QMDD for 3-variable Majority Function



Matrix Multiplication

$$\begin{bmatrix} A_0 & A_1 & \cdots & A_{r-1} \\ A_r & A_{r+1} & \cdots & A_{2r-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_{r^2-r} & A_{r^2-r+1} & \cdots & A_{r^2-1} \end{bmatrix} \times \begin{bmatrix} B_0 & B_1 & \cdots & B_{r-1} \\ B_r & B_{r+1} & \cdots & B_{2r-1} \\ \vdots & \vdots & \ddots & \vdots \\ B_{r^2-r} & B_{r^2-r+1} & \cdots & B_{r^2-1} \end{bmatrix}$$

$$= \begin{bmatrix} A_0B_0 + A_1B_r + \ldots + A_{r-1}B_{r^2-r} & A_0B_1 + A_1B_{r+1} + \ldots + A_{r-1}B_{r^2-r+1} & \cdots & \vdots \\ A_rB_0 + A_{r+1}B_r + \ldots + A_{2r-1}B_{r^2-r} & \vdots & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \cdots & \cdots & \cdots & A_{r^2-r}B_{r-1} + A_{r^2-r+1}B_{2r-1} + \ldots + A_{r^2-1}B_{r^2-1} \end{bmatrix}$$

NOTE: Handling of “skipped variables” differs for matrices and vectors.

QMDD-based Spectral Transformation

The QMDD package is written in C and was designed for work related to reversible and quantum circuits.

Extending it to handle spectral transformation computation was reasonably straightforward.

1. Implementing a routine to build a QMDD given the truth vector for a function. This is a simple recursive construction.

2. Implementing a routine to build the QMDD for the required transformation matrix. This is a simple iterative routine as a result of the compact linear structure of the QMDD for transformations defined as the Kronecker product of a single basis matrix.

3. Modifications to the matrix multiplication and addition routines to handle null pointers and general skipped variables separately for matrices and column vectors.

4. Modification to the complex value operation routines to do computation over GF(2) when required for the transformation.

Experimental Results

Table 1: Results for AND function – Rademacher-Walsh Transform (+1,-1) coding.

n	F size	S size	Edge Weights	Add	Mult	Total	FT ops	Total / FT ops	(a) QMDD size	(b) FT size	(a) / (b)
2	3	3	5	12	17	29	8	362.50%	212	16	1325.00%
3	4	7	8	32	33	123	24	512.50%	436	32	1362.50%
4	5	10	14	60	49	181	64	282.81%	664	64	1037.50%
5	6	13	20	96	65	181	160	113.13%	892	128	696.88%
6	7	16	26	142	85	241	384	62.76%	1120	256	437.50%
7	8	19	32	192	97	315	896	35.16%	1348	512	263.28%
8	9	22	38	252	113	397	2048	19.38%	1576	1024	153.91%

Table 2: Results for OR function – Rademacher-Walsh Transform (+1,-1) coding.

n	F size	S size	Edge Weights	Add	Mult	Total	FT ops	Total / FT ops	(a) QMDD size	(b)FT size	(a) / (b)
2	3	3	5	12	17	29	8	362.50%	212	16	1325.00%
3	4	4	9	26	29	55	24	229.17%	320	32	1000.00%
4	5	5	13	44	41	85	64	132.81%	428	64	668.75%
5	6	6	17	66	53	119	160	74.38%	536	128	418.75%
6	7	7	21	92	65	157	384	40.89%	644	256	251.56%
7	8	8	25	122	77	199	896	22.21%	752	512	146.88%
8	9	9	29	156	89	245	2048	11.96%	860	1024	83.98%

Table 3: Results for XOR function – Rademacher-Walsh Transform (+1,-1) coding.

n	F size	S size	Edge Weights	Add	Mult	Total	FT ops	Total / FT ops	(a) QMDD size	(b) FT size	(a) / (b)
2	4	3	6	8	17	25	8	312.50%	228	16	1425.00%
3	6	4	8	16	33	49	24	204.17%	304	32	950.00%
4	8	5	10	24	49	73	64	114.06%	380	64	593.75%
5	10	6	12	32	65	97	160	60.63%	456	128	356.25%
6	12	7	14	40	81	121	384	31.51%	532	256	207.81%
7	14	8	16	48	97	145	896	16.18%	608	512	118.75%
8	16	9	18	56	113	169	2048	8.25%	684	1024	66.80%

Table 4: Results for random functions – Rademacher-Walsh Transform (+1,-1) coding.

n	F size	S size	Edge Weights	Add	Mult	Total	FT ops	Total / FT ops	(a) QMDD size	(b) FT size	(a) / (b)
2	4	3	6	8	17	25	8	312.50%	228	16	1425.00%
3	6	7	8	34	37	71	24	295.83%	436	32	1362.50%
4	9	11	13	74	77	151	64	235.94%	692	64	1081.25%
5	15	21	17	258	161	419	160	261.88%	1196	128	934.38%
6	25	47	30	720	413	1133	384	295.05%	2548	256	995.31%
7	43	96	76	1794	773	2567	896	286.50%	5440	512	1062.50%
8	74	196	130	4458	1385	5843	2048	285.30%	10704	1024	1045.31%

Table Key
n – number of variables
F size – number of vertices in QMDD for the function
S size – number of vertices in QMDD for the spectrum
Edge Weights – number of unique edge weights
Add – number of add operations in QMDD transform
Mult – number of multiplication operations in QMDD transform
Total – total operations in QMDD transform
FT ops – operations in fast transform
QMDD size – size of QMDD for spectrum (bytes)
FT size – size of spectrum vector for fast transform (bytes)

Table 5: Comparisons for various functions and transformations.

		n	F size	S size	Edge Weights	Add	Mult	Total	FT ops	(a) QMDD size	((b) FT size	(a) / (b)
S	AND	15	16	43	80	916	261	2.15E+08	0.00%	3172	131072	2.42%
RM	AND	15	16	16	2	30	61	2.15E+08	0.00%	736	131072	0.56%
R	AND	15	16	30	3	58	117	2.15E+08	0.00%	1368	131072	1.04%
S	OR	15	16	16	57	506	173	2.15E+08	0.00%	1616	131072	1.23%
Rm	OR	15	16	16	2	268	117	2.15E+08	0.00%	736	131072	0.56%
R	OR	15	16	16	59	506	173	2.15E+08	0.00%	1648	131072	1.26%
S	XOR	15	30	16	32	112	225	2.15E+08	0.00%	1216	131072	0.93%
RM	XOR	15	30	30	2	450	117	2.15E+08	0.00%	1352	131072	1.03%
R	XOR	15	30	30	31	320	225	2.15E+08	0.00%	1816	131072	1.39%
S	random	12	736	2596	1084	283446	50161	6377292	5.23%	131568	16384	803.03%
RM	random	12	736	724	2	17248	3293	6377292	0.32%	31888	16384	194.63%
R	random	12	736	2597	1442	299568	47633	6377292	5.44%	137340	16384	767.85%

Table 6: Results for MIN function r=3 – Chrestenson Spectrum.

n	F size	S size	Edge Weights	Add	Mult	Total	FT ops	Total / FT ops	(a) QMDD size	(b) FT size	(a) / (b)
2	4	5	27	66	73	139	18	772.22%	852	144	591.67%
3	6	11	66	237	154	391	81	482.72%	1980	432	458.33%
4	8	20	122	681	397	1078	324	332.72%	3632	1296	280.25%
5	10	32	191	1152	316	1468	1215	120.82%	5744	3888	147.74%
6	12	47	257	1980	397	2377	4374	54.34%	8060	11664	69.10%
7	14	65	323	3120	559	3679	15309	24.03%	10628	34992	30.37%
8	16	86	389	4440	559	4999	52488	9.52%	13448	104976	12.81%
9	18	110	455	6138	640	6778	177147	3.83%	16520	314928	5.25%
10	20	137	521	8214	874	9088	590490	1.54%	19844	944784	2.10%

Table 7: Results for MAX function r=3 – Chrestenson Spectrum.

n	F size	S size	Edge Weights	Add	Mult	Total	FT ops	Total / FT ops	(a) QMDD size	(b) FT size	(a) / (b)
2	4	5	28	66	73	139	18	772.22%	868	144	602.78%
3	6	11	64	192	136	328	81	404.94%	1948	432	450.93%
4	8	20	118	387	199	586	324	180.86%	3568	1296	275.31%
5	10	32	174	672	262	934	1215	76.87%	5472	3888	140.74%
6	12	47	230	1047	325	1372	4374	31.37%	7628	11664	65.40%
7	14	65	286	1521	388	1909	15309	12.47%	10036	34992	28.68%
8	16	86	341	2076	451	2527	52488	4.81%	12680	104976	12.08%
9	18	110	398	2736	514	3250	177147	1.83%	15608	314928	4.96%
10	20	137	454	3498	577	4075	590490	0.69%	18772	944784	1.99%

Table 8: Results for MOD-SUM function r=3 – Chrestenson Spectrum.

n	F size	S size	Edge Weights	Add	Mult	Total	FT ops	Total / FT ops	(a) QMDD size	(b) FT size	(a) / (b)
2	5	3	20	48	73	121	18	672.22%	572	144	397.22%
3	8	4	29	102	154	256	81	316.05%	800	432	185.19%
4	11	5	38	156	235	391	324	120.68%	1028	1296	79.32%
5	14	6	47	210	316	526	1215	43.29%	1256	3888	32.30%
6	17	7	56	264	397	661	4374	15.11%	1484	11664	12.72%
7	20	8	65	318	478	796	15309	5.20%	1712	34992	4.89%
8	23	9	74	372	559	931	52488	1.77%	1940	104976	1.85%

Concluding Remarks

- The method is a consistent approach to a variety of transformations for binary and multiple-valued functions.
- It is most efficient for transformations composed as a Kronecker product of an
  $r \times r$ base matrix.
- Formal analysis of the complexity of the approach is complicated by the use of computed and computation tables in the QMDD implementation.
- Ongoing work involves application of this approach for benchmark examples and studying the effect of variable reordering on the QMDD size.

Acknowledgment

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