The Shape of the SNF as a Source of Information

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- SNF Based Cube Detection
  - Adjacency Graph of a SNF
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- Complexity Evaluated by the SNF
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Introduction

- Sum-Of-Products (SOP)

$$f(a,b,c) = \overline{a}\,\overline{b} \vee \overline{a}\,\overline{c} \vee abc$$

  - well studied

- Exclusive Sum-Of-Products (ESOP)

$$f(a,b,c) = \overline{a} \oplus bc$$

  - more compact than the SOP, in general
  - less studied

c					f
0	1	1	0	0	
1	1	0	1	0	
	0	1	1	0	b
	0	0	1	1	a

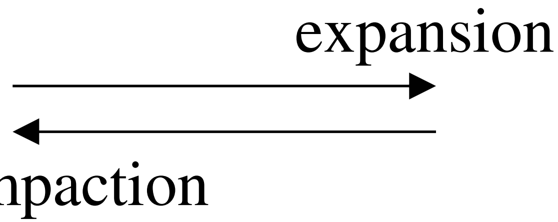
Introduction

- algebraic property of the exclusive-or operation

$$x = \overline{x} \oplus 1 \qquad (1)$$

$$\overline{x} = 1 \oplus x \qquad (2)$$

$$1 = x \oplus \overline{x} \qquad (3)$$



- isomorphic properties of the set $\{x, \overline{x}, 1\}$

Introduction

- Algorithm 1 Calculate Exp(f)

Require: any ESOP of a Boolean function f

Ensure: complete expansion of the Boolean function f w.r.t. all variables of its support

1: for all variables V_i of the support of f do

2: for all cubes C_j of f do

3: $\langle C_{n1}, C_{n2} \rangle \leftarrow$ **expand**(C_j , V_i)

4: replace C_j by $\langle C_{n1}, C_{n2} \rangle$

5: end for

6: end for

Introduction

- property of the exclusive-or operation for a Boolean function f

$$f = f \oplus 0 \qquad (4)$$

$$0 = C \oplus C \qquad (5)$$

$$f = f \oplus C \oplus C \qquad (6)$$

reduction ←

- pairs of cubes can be removed from the ESOP without change of the Boolean function f

Introduction

- Algorithm 2 Calculate R(f)

Require: any ESOP of a Boolean function f containing n cubes
Ensure: reduced ESOP of f containing no cube more than once

for $i \leftarrow 0$ to $n - 2$ do
    for $j \leftarrow i + 1$ to $n - 1$ do
        if $C_i = C_j$ then
            $C_i \leftarrow C_{n-1}$
            $C_j \leftarrow C_{n-2}$
            $n \leftarrow n - 2$
            $j \leftarrow i$
        end if
    end for
end for

Introduction

- Definition 1 - SNF(f)

  Take any ESOP of a Boolean function f. The resulting ESOP of

  $$SNF(f) = R(Exp(f)) \qquad (7)$$

  is called Specialized Normal Form (SNF) of the Boolean function.

- Exp(f) distributes the information about cubes
- R(f) removes pairs of cubes
- The SNF(f) is a unique ESOP of f.

SNF Based Cube Detection – $AG^{SNF(f)}$

- Definition 2 - Hamming distance δ

  The Hamming distance between two cubes $x_1, x_2, \ldots, x_n$, and $y_1, y_2, \ldots, y_n$, is the number of unequal (ternary) coordinates.

- Definition 3 - Adjacency Graph $AG^{SNF(f)}(V,E)$

  The vertices V of the adjacency graph $AG^{SNF(f)}(V,E)$ correspond to the cubes of the SNF(f). Each vertex carries the ternary vector of the associated cube as label. Two vertices V of $AG^{SNF(f)}(V,E)$ are connected by an edge, if they have a Hamming distance δ equal to one.

SNF Based Cube Detection – $AG^{SNF(f)}$

- Example

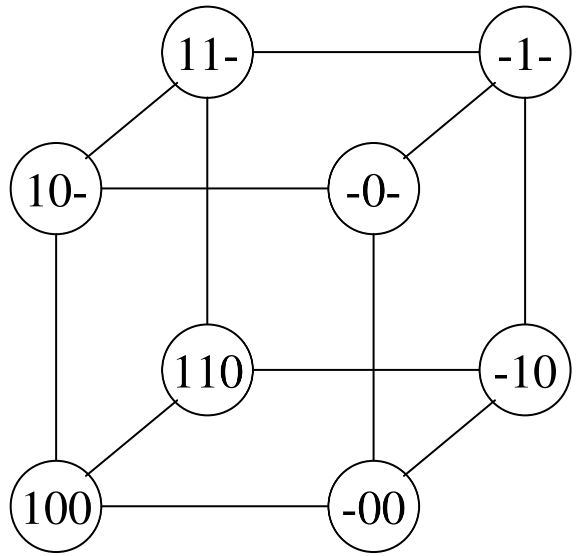
$$f(a,b,c) = \overline{a}c$$

	a	b	c
f =	0	-	1

SNF(f) =

a	b	c
1	0	0
1	0	-
1	1	0
1	1	-
-	0	0
-	0	-
-	1	0
-	1	-

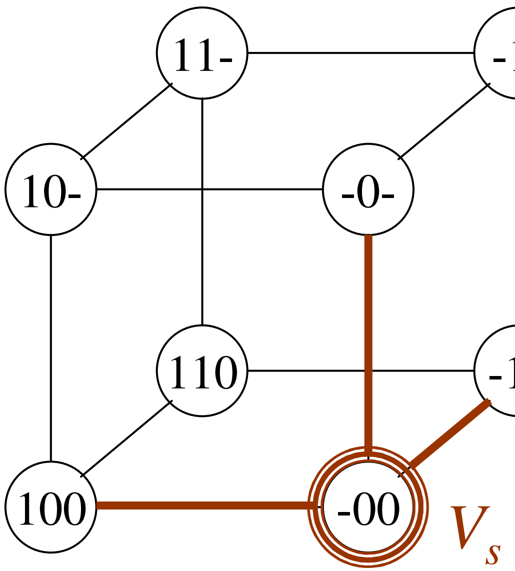
$AG^{SNF(f)}(V,E)$



SNF Based Cube Detection – $AG^{SNF(f)}$

- Reconstruction of the cube $f(a,b,c) = \overline{a}c$

$AG^{SNF(f)}(V,E)$



a	b	c	a	b	c		a	b	c
-	0	0	1	0	0	→	0		
-	0	0	-	1	0	→		-	
-	0	0	-	0	-	→			1
							0	-	1

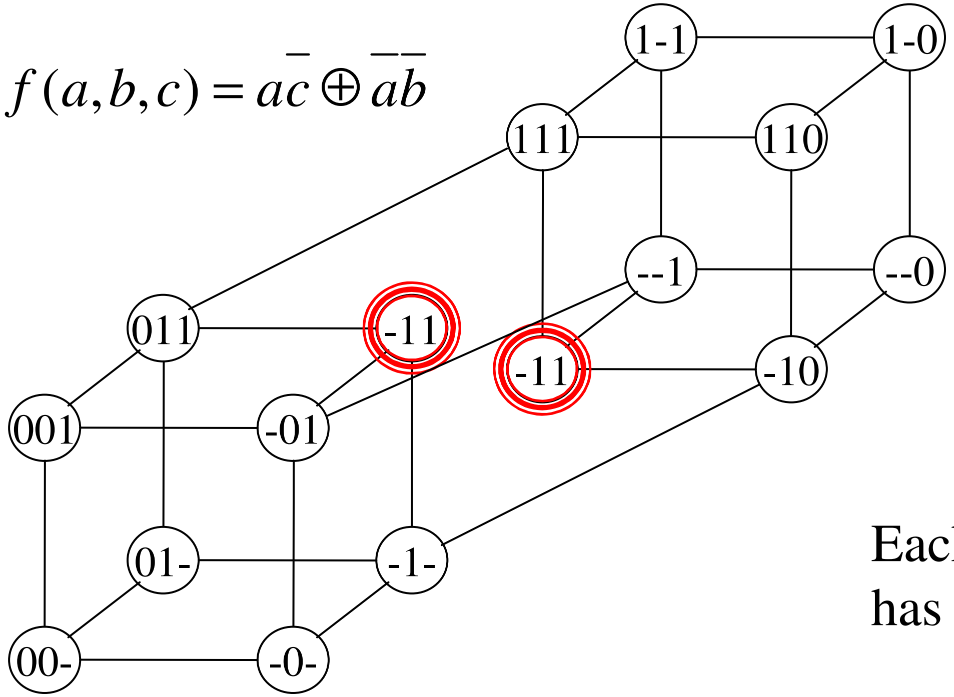
HCCC(AG, V_s).
hypercube corner compaction

SNF Based Cube Detection – $AG^{SNF(f)}$

- Definition 4 - Degree D of the vertex

  The degree D of vertex in a graph is the number of other vertices connected by edges directly to basic vertex.

$f(a,b,c) = a\bar{c} \oplus \overline{a}\overline{b}$



The adjacency graph $AG^{SNF(f)}(V,E)$ for a Boolean function $f: B^n \rightarrow B$ is a *n*-regular graph.

Each vertex in an *n*-regular graph has a degree of D = *n*.

Adjacency Graph of a SNF

- all reconstructed cubes of the adjacency graph

selected cube	neighbor cube 1	neighbor cube 2	neighbor cube 3	Created cube	associated conjunction
00–	–0–	01–	001	1–0	$a\bar{c}$
–0–	00–	–1–	–01	1–0	$a\bar{c}$
01–	–1–	00–	011	1–0	$a\bar{c}$
001	–01	011	00–	1–0	$a\bar{c}$
011	111	001	01–	––0	$\bar{c}$
–01	001	––1	–0–	110	$ab\bar{c}$
–1–	01–	–0–	–10	1–1	ac
–10	110	––0	–1–	001	$\bar{a}\bar{b}c$
––1	1–1	–01	––0	01–	$\bar{a}b$
111	011	1–1	110	–0–	b
––0	1–0	–10	––1	00–	$\bar{a}\bar{b}$
110	–10	1–0	111	00–	$\bar{a}\bar{b}$
1–1	–11	111	1–0	00–	$\bar{a}\bar{b}$
1–0	––0	110	1–1	00–	$\bar{a}\bar{b}$

SNF Based Cube Detection – $AG^{SNF(f)}$

- Definition 5 - Distance-one wrapper cubes of SNF(f)

  Each cube from the same Boolean space like f that does not belong to SNF(f), but has a Hamming distance of δ equal to one to at least cube of the SNF(f) is a distance-one wrapper cube of the SNF(f) .

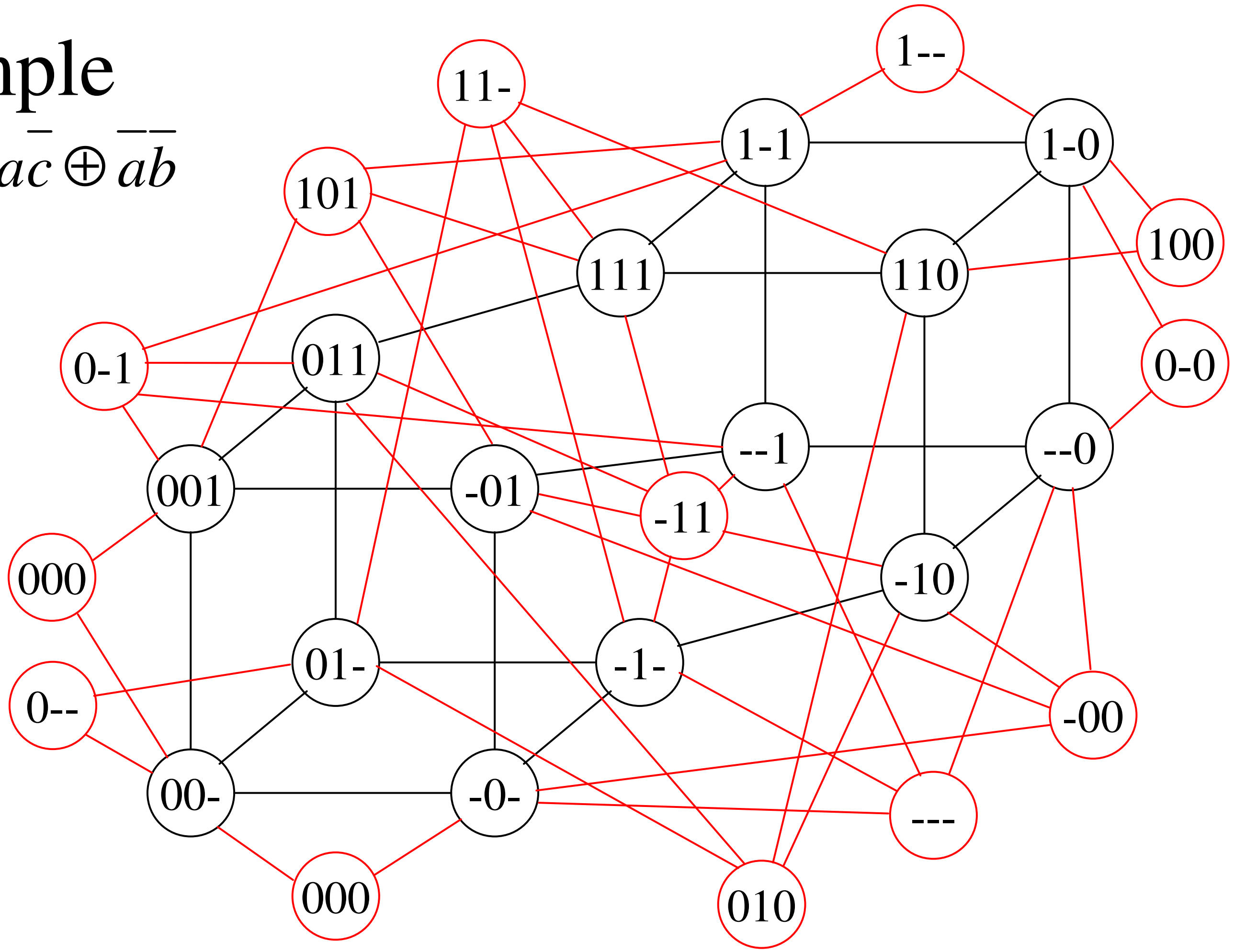
- Definition 6 - Extended adjacency graph $EAG^{SNF(f)}$ (V,E) of the SNF(f)

  The extended adjacency graph of the SNF(f) consists of the adjacency graph $AG^{SNF(f)}$ (V,E) of the SNF(f) as core, extended by vertices of all distance-one wrapper cubes of the SNF(f) and edges between these wrapper vertices and the core vertices of the SNF cubes having a Hamming distance of δ equal to one. There are no edges between the wrapper vertices.

SNF Based Cube Detection – EAG$^{SNF(f)}$

- Example

$$f(a,b,c) = a\bar{c} \oplus \overline{a}\,\overline{b}$$



SNF Based Cube Detection – Weight

- Algorithm 3 –

  Calculate Weights for the vertices of $\text{EAG}^{\text{SNF}(f)}\,(V,E)$

Require: extended adjacency graph $\text{EAG}^{\text{SNF}(f)}(V,E)$ of a Boolean function f
Ensure: weights of all vertices of $\text{EAG}^{\text{SNF}(f)}(V,E)$
1: for all wrapper vertices $V_w[i]$ of $\text{EAG}^{\text{SNF}(f)}(V,E)$ do
2: $\quad$ $weight(V_w[i]) \leftarrow \texttt{degree}(V_w[i])$
3: end for
4: for all core vertices $V_c[j]$ of $\text{EAG}^{\text{SNF}(f)}(V,E)$ do
5: $\quad$ $weight(V_c[j]) \leftarrow 0$
6: $\quad$ for all adjacent wrapper vertices $V_w[i]$ of $\text{EAG}^{\text{SNF}(f)}(V,E)$ do
7: $\quad\quad$ $weight(V_c[j]) \leftarrow weight(V_c[j]) + weight(V_w[i])$
8: $\quad$ end for
9: end for

SNF Based Cube Detection – Weight

- selection of core vertex by its weight in extended adjacency graph

vertex

00–	–0–	01–	001	011	–01	–1–	–10	––1	111	––0	110	1–1	1–0
6	10	10	10	14	14	14	14	14	14	10	10	10	6

weights of the core vertices
in extended adjacency graph

SNF Based Cube Detection

- Definition 7 - Edge distance Δ

  The edge distance Δ between two vertices of an SNF is the minimum number of edges in the associated adjacency graph we have to travel to get from the first to the second vertex.

- Definition 8 - Hamming degree d_k

  The k th Hamming degree d_k of a cube of an SNF is the number of vertices of the associated adjacency graph which are at a Hamming distance δ equal to k.

- Definition 9 - Edge degree D_k

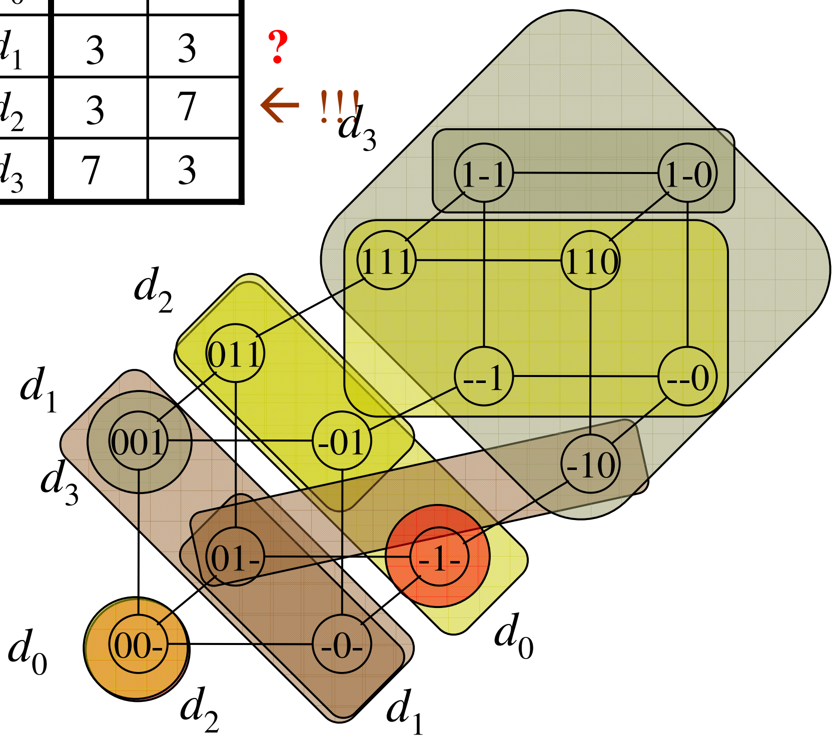
  The k th edge degree D_k of a cube of an SNF is the number of vertices of the associated adjacency graph which are at an edge distance Δ equal to k.

SNF Based Cube Detection

- Hamming degree d_k

	00-	-1-
d_0	1	1
d_1	3	3
d_2	3	7
d_3	7	3

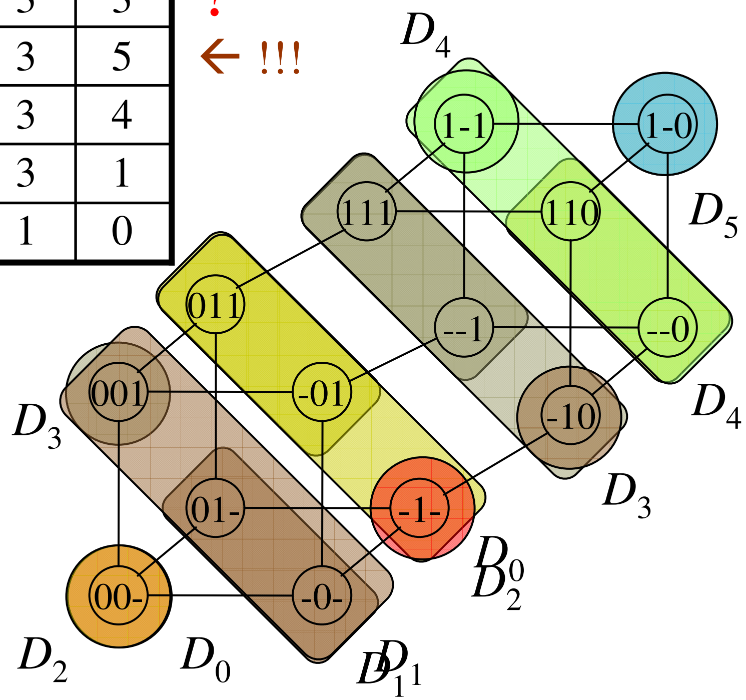
? ← !!!



- Edge degree D_k

	00-	-1-
D_0	1	1
D_1	3	3
D_2	3	5
D_3	3	4
D_4	3	1
D_5	1	0

? ← !!!



SNF Based Cube Detection

Table 3: Hamming degrees d_k of the SNF(f) of Figure 2

	00-	-0-	01-	001	011	-01	-1-	-10	--1	111	--0	110	1-1	1-0	sum
d_0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	14
d_1	3	3	3	3	3	3	3	3	3	3	3	3	3	3	42
d_2	3	5	5	5	7	7	7	7	7	7	5	5	5	3	78
d_3	7	5	5	5	3	3	3	3	3	3	5	5	5	7	62
sum	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14^2

Table 4: Edge degrees D_k of the SNF(f) of Figure 2

	00-	-0-	01-	001	011	-01	-1-	-10	--1	111	--0	110	1-1	1-0	sum
D_0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	14
D_1	3	3	3	3	3	3	3	3	3	3	3	3	3	3	42
D_2	3	4	4	4	5	5	5	5	5	5	4	4	4	3	78
D_3	3	4	4	4	4	4	4	4	4	4	4	4	4	3	78
D_4	3	2	2	2	1	1	1	1	1	1	2	2	2	3	14
D_5	1	0	0	0	0	0	0	0	0	0	0	0	0	1	14
sum	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14^2

Complexity Evaluated by the SNF

- B^1

```
# ALLBF = 4
# SNF = 0     # EMIN = 0   # BF 1
# SNF = 2     # EMIN = 1   # BF 3
```

- simplest function: $f(x) = 0$
- most complex functions: $f(x) = x, \quad f(x) = \overline{x}, \quad f(x) = 1$
- number of classes of functions: 2
- number of cubes in most complex SNFs: $n_{\max}^{SNF}(n) = 2 * 3^{n-1}$

  $n_{\max}^{SNF}(1) = 2 * 3^{1-1} = 2$
- number of most complex functions: $n_{\max}^{BF}(n) = 3 * 2^{n-1}$

  $n_{\max}^{BF}(1) = 3 * 2^{1-1} = 3$

Complexity Evaluated by the SNF

- B^2

```
# ALLBF = 16
# SNF = 0        # EMIN = 0   # BF 1
# SNF = 4        # EMIN = 1   # BF 9
# SNF = 6        # EMIN = 2   # BF 6
```

2^2

3^2

- Hamming distance δ of most complex functions: 2
- number of classes of functions: 3
- number of cubes in most complex SNFs: $n_{\max}^{SNF}(n) = 2 * 3^{n-1}$

  $n_{\max}^{SNF}(2) = 2 * 3^{2-1} = 6$

- number of most complex functions: $n_{\max}^{BF}(n) = 3 * 2^{n-1}$

  $n_{\max}^{BF}(2) = 3 * 2^{2-1} = 6$

Complexity Evaluated by the SNF

- B^3

```
# ALLBF = 256
# SNF = 0       # EMIN = 0   # BF 1
# SNF = 8       # EMIN = 1   # BF 27
# SNF = 12      # EMIN = 2   # BF 54
# SNF = 14      # EMIN = 2   # BF 108
# SNF = 16      # EMIN = 3   # BF 54
# SNF = 18      # EMIN = 3   # BF 12
```

2^3

3^3

δ=2, 6*3*3 = 54

δ=3,
$3^3 * 2^3 / 2 = 108$

δ	1	2	3
1	0	2	3
2	2	0	3
3	3	3	0

δ	1	2	3
1	0	3	3
2	3	0	3
3	3	3	0

- number of classes of functions: 5
- number of cubes in most complex SNFs: $n_{\max}^{SNF}(3) = 2 * 3^{3-1} = 18$
- number of most complex functions: $n_{\max}^{BF}(3) = 3 * 2^{3-1} = 12$

Complexity Evaluated by the SNF

- B^4

2^4 3^4

ALLBF = 65536

# SNF = 0	# EMIN = 0	# BF 1
# SNF = 16	# EMIN = 1	# BF 81
# SNF = 24	# EMIN = 2	# BF 324
# SNF = 28	# EMIN = 2	# BF 1296
# SNF = 30	# EMIN = 2	# BF 648
# SNF = 32	# EMIN = 3	# BF 648
# SNF = 34	# EMIN = 3	# BF 3888
# SNF = 36	# EMIN = 3	# BF 6624
# SNF = 36	# EMIN = 4	# BF 108
# SNF = 38	# EMIN = 3	# BF 7776
# SNF = 40	# EMIN = 3	# BF 2592
# SNF = 40	# EMIN = 4	# BF 6642
# SNF = 42	# EMIN = 3	# BF 216
# SNF = 42	# EMIN = 4	# BF 14256
# SNF = 44	# EMIN = 4	# BF 12636
# SNF = 46	# EMIN = 4	# BF 3888
# SNF = 46	# EMIN = 5	# BF 1296
# SNF = 48	# EMIN = 5	# BF 1944
# SNF = 50	# EMIN = 5	# BF 648
# SNF = 54	# EMIN = 6	# BF 24

The number of different sizes of the SNF for minimal ESOPs of the same number of cubes grows, such that for Boolean function of 4 and more variables these intervals overlap

The total number of cubes in the SNF depends on both, the number of cubes in the minimal ESOP and their Hamming distance δ.

$$n_{\max}^{SNF}(4) = 2 * 3^{4-1} = 54$$

$$n_{\max}^{BF}(4) = 3 * 2^{4-1} = 24$$

Experimental Results

- the new observation about the Hamming degree d_2 was verified calculating all exact minimal ESOPs of B^4

- calculation of all exact minimal ESOPs of B^4

	time in seconds		ratio
weight	959		184%
d_2		1767	
all	2186	2994	137%
ratio	44%	59%	

- time for each cube of the SNF:
  - weight O(n)
  - Hamming degree d_2 O(n^2)

Conclusion

- knowledge about the application of the specialized normal form SNF(f) could be extended for:
  1. SNF based cube detection
  2. SNF based evaluation of the complexity of logic functions

Conclusion

- SNF based cube detection
  - the basic approach requires distance-1 wrapper cubes in order to find minimal weights of core vertices in the extended adjacency graph $EAG^{SNF(f)}(V,E)$
  - alternatively the Hamming degree d_k and the edge degree D_k are suggested
  - experimental results for B^4 verified that the application of the Hamming degree d_2 leads to the same known results of all exact minimal ESOPs
  - the benefit of the new approach is that the second degrees can be calculated directly based on the SNF without additional distance-1 wrapper cube
  - a drawback of d_2 is the complexity $O(n^2)$ instead of $O(n)$ for the weights

Conclusion

- SNF based evaluation of the complexity of logic functions
  - the classification of Boolean functions based on the number of cubes in their SNFs and the number of cubes in their exact minimal ESOPs was extended by the Hamming distances δ between the cubes in the minimal ESOP
  - this new category allows to distinguish the complexity for minimal ESOPs having the same number of cubes
  - this observation may be a root for further theoretical results