The Attractor Structure of Rule 60 Cellular Automata

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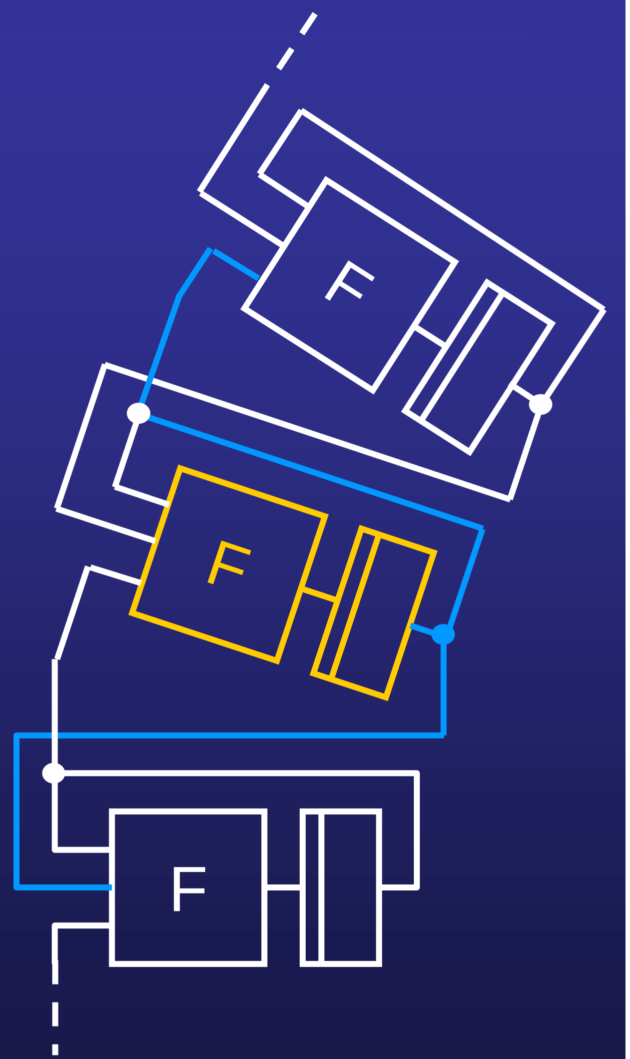
- Terminology
- Results on attractor length
- CA state space constructed from a small fragment

Cellular Automata

apparently?

- Complex behavior by simple local computation
- Regular communication structure

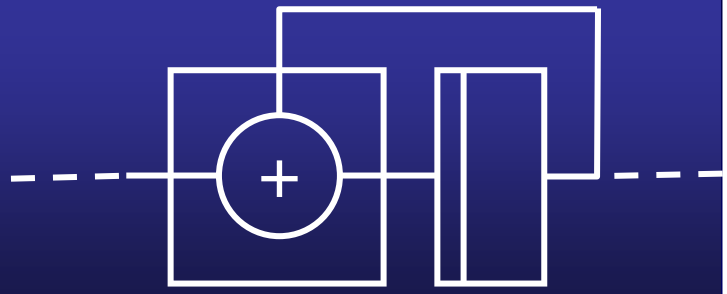


Motivation and aims

- Circular CA as a test sequence generator
- With random initial state, properties similar to other generator types
- With initial state of low Hamming weight, vastly differing behavior

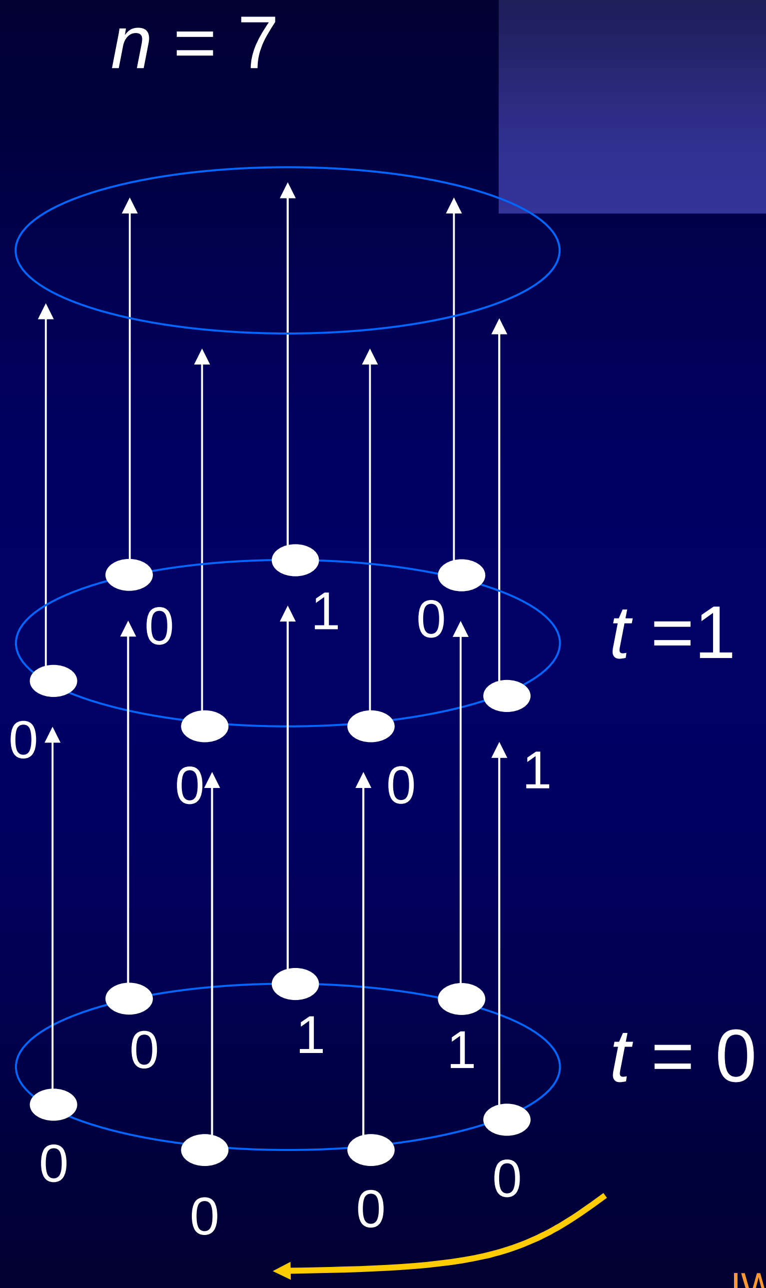
•to understand
•to generate (a superset of) given output
•to control average weights of CA outputs

Why to like Rule 60 automata



- For mathematicians:
  F is additive (linear);
  analysis:
  - linear algebra, eigenvalues
  - (di)polynomials (Wolfram 1984)
- For engineers:
  it is a ring of T flip-flops;
  small silicon area,
  simple interconnection

A spatial image of the state space

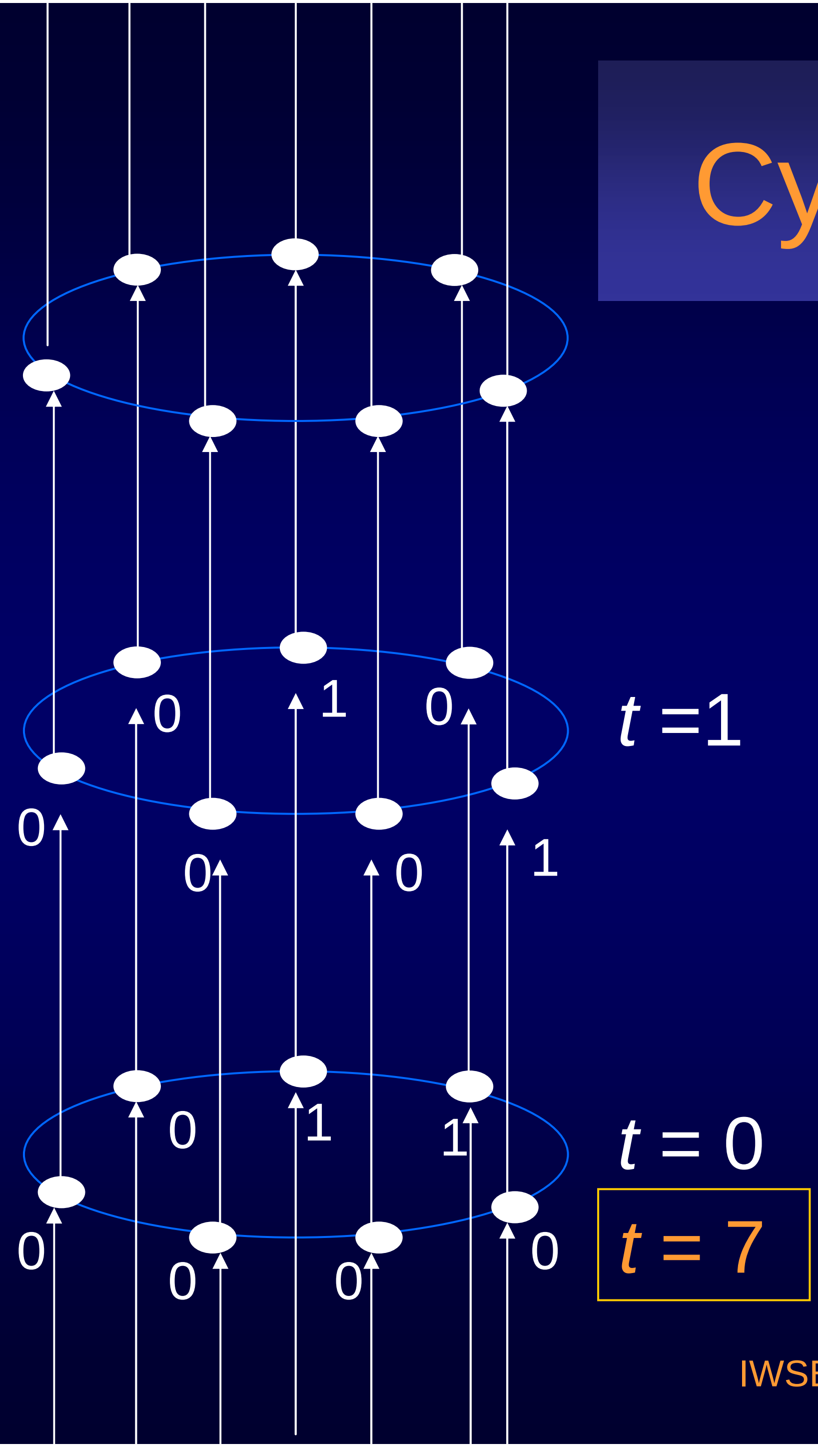


- A circle has no beginning
- But a CA60 ring has orientation
- States at t = 1, 2, … form a cylinder
- Hence “cylindrical CA”

A circle has no beginning

- All propositions about CA with circular structure must be invariant to rotation
- Can be generalized to any symmetry of the CA structure
- “Symmetry kills randomness”

Cyclic behavior

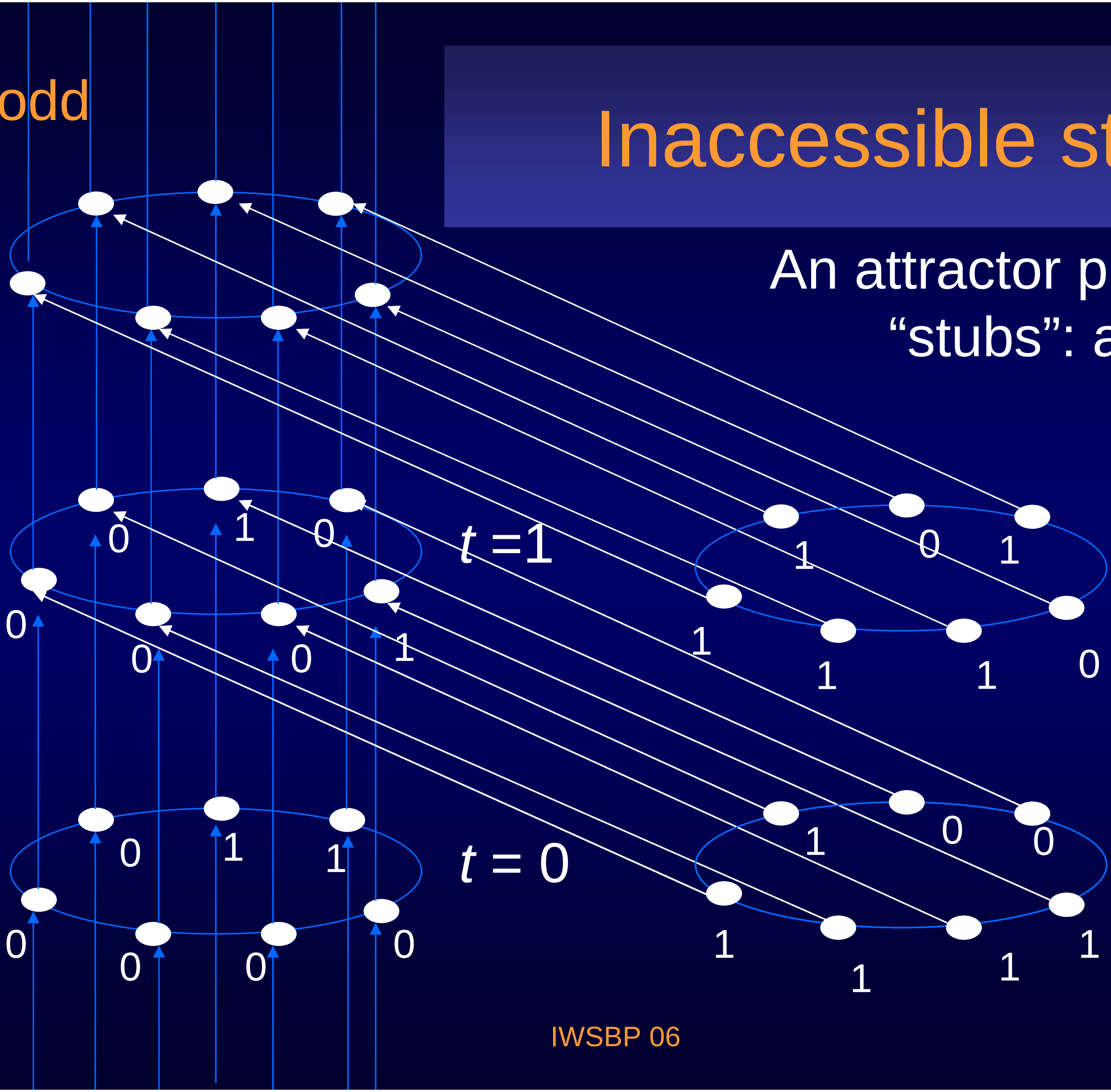


- the “cylinder” is a torus
- the state space comprises multiple tori
- the cycles are *attractors*

Inaccessible states

n odd

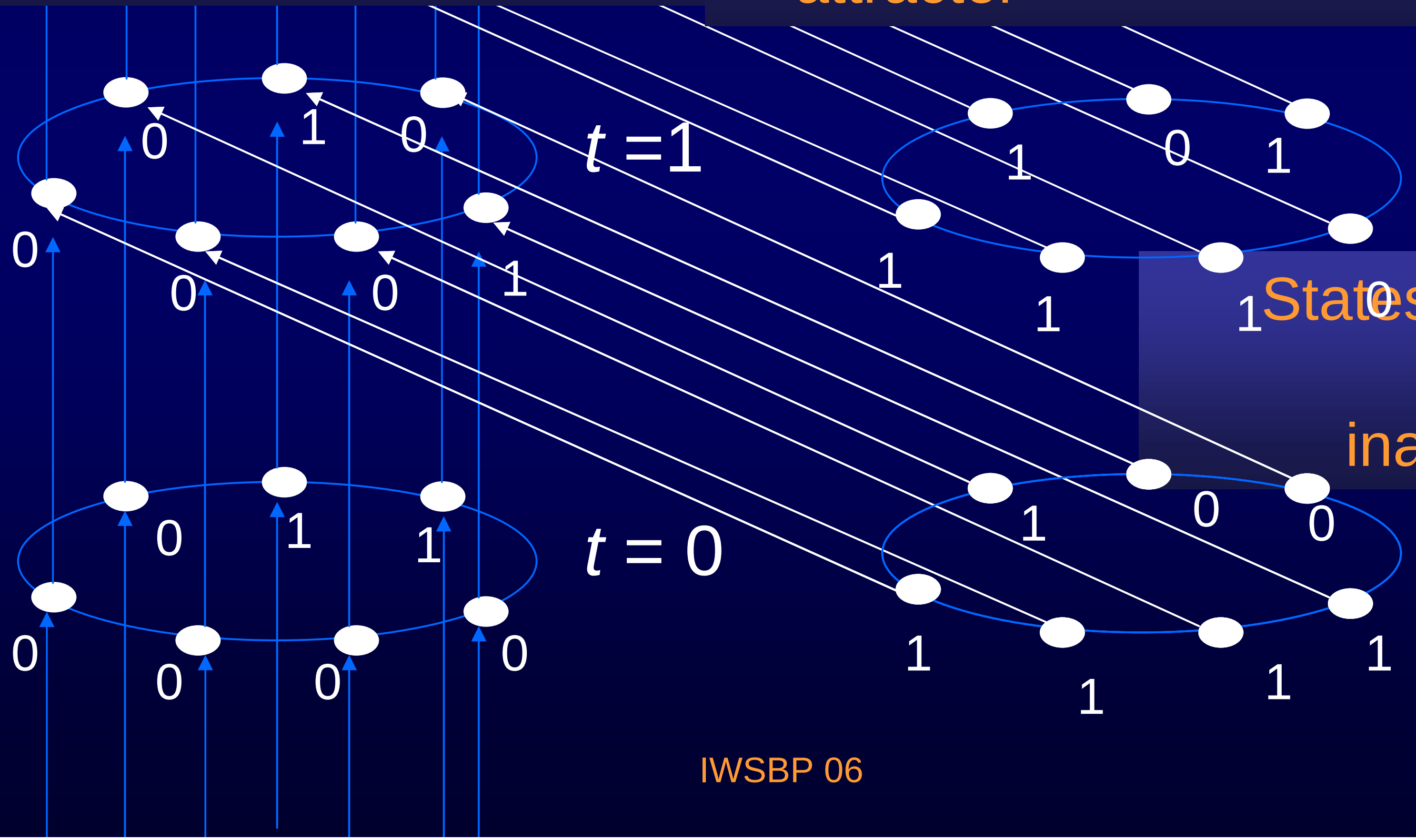
An attractor plus the "stubs": a *basin*



All CA with n odd, 2 inputs to F

Each accessible state has two predecessors with mutually inverse values

All descendants of an inaccessible state lie on an attractor



States with odd parity are inaccessible

An outline of the state space

- States with odd parity are inaccessible
- One half of all spaces is inaccessible
- All descendants of an inaccessible state lie on an attractor
- Each accessible state has two predecessors
- These predecessors have mutually inverse values

Holds for all CA with *n* odd,
F having two inputs only

Attractor lengths

- Two important functions:
  - order: $\mathrm{ord}_n(2) = j \Leftrightarrow 2^j \equiv 1 \bmod n,\ j$ min
  - suborder: $\mathrm{sord}_n(2) = j \Leftrightarrow 2^j \equiv -1 \bmod n,\ j$ min
- Properties: either
  - $\mathrm{ord}_n(2) = \mathrm{sord}_n(2)$, or ←Type A
  - $\mathrm{ord}_n(2) = 2\,\mathrm{sord}_n(2)$ ←Type B
- Attractor lengths:
  - $L^A_n = 2^{\mathrm{sord}_n(2)} - 1$ ←Type A
  - $L^B_n = n\,(2^{\mathrm{sord}_n(2)} - 1)$ ←Type B

Consequences

- Type B more frequent for large *n*
- Exponentially many attractors, approximately

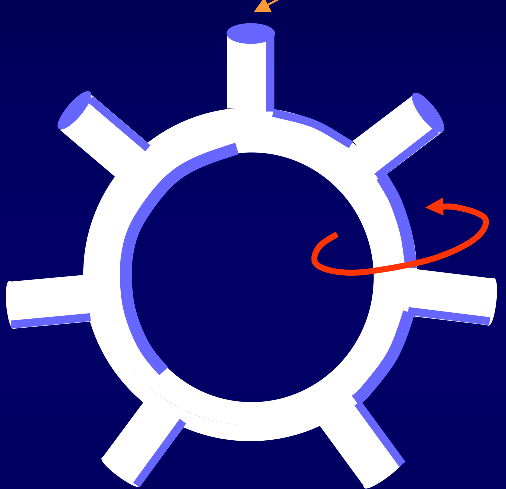
  $A_n = n^{-1}.2^n$

- The first *n* steps from a state with a single ‘1’ has different output weight structure; this may be the entire attractor or a negligible part depending on *n*

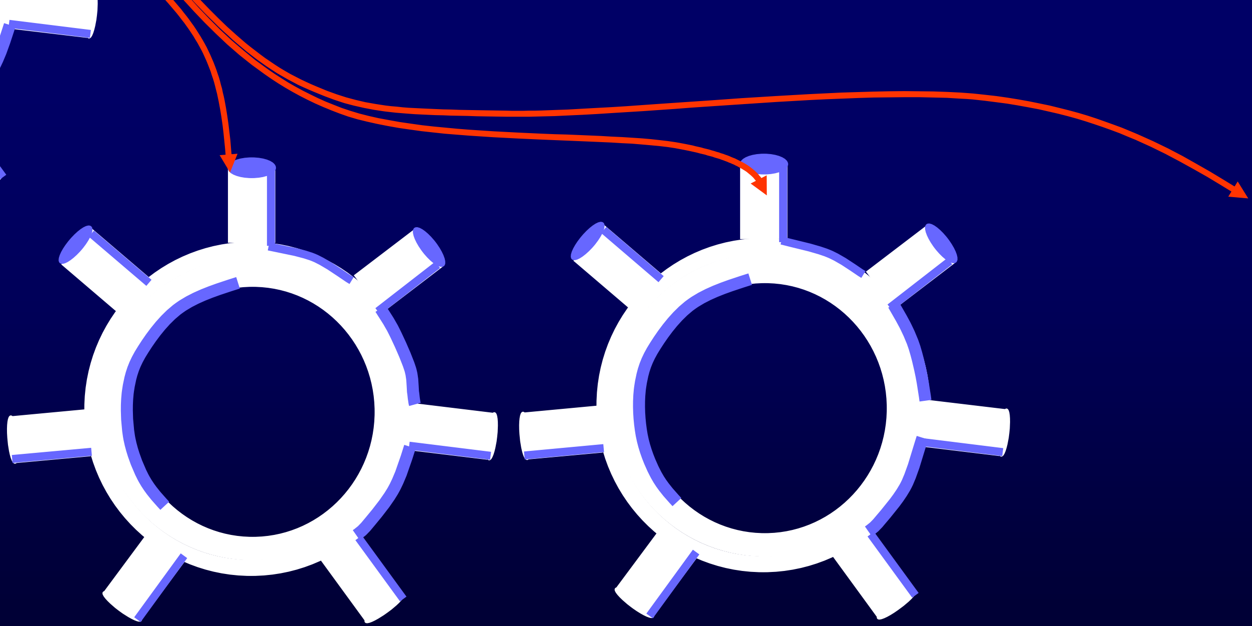
Type A CA60: primary basins

n = 7

a state with only one ‘1’

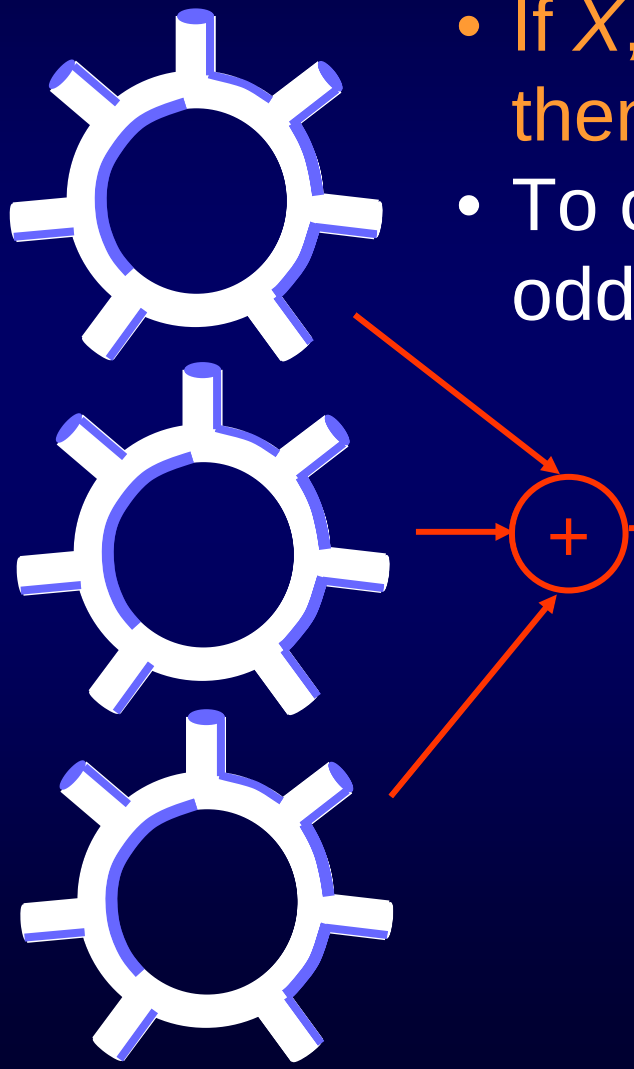


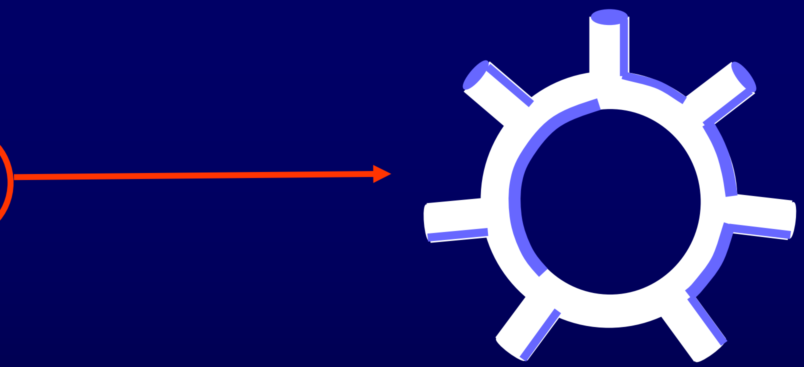
a circle has no beginning ⇒ *n* such inaccessible states must exist; here: in different basins



Type A CA60: basin addition

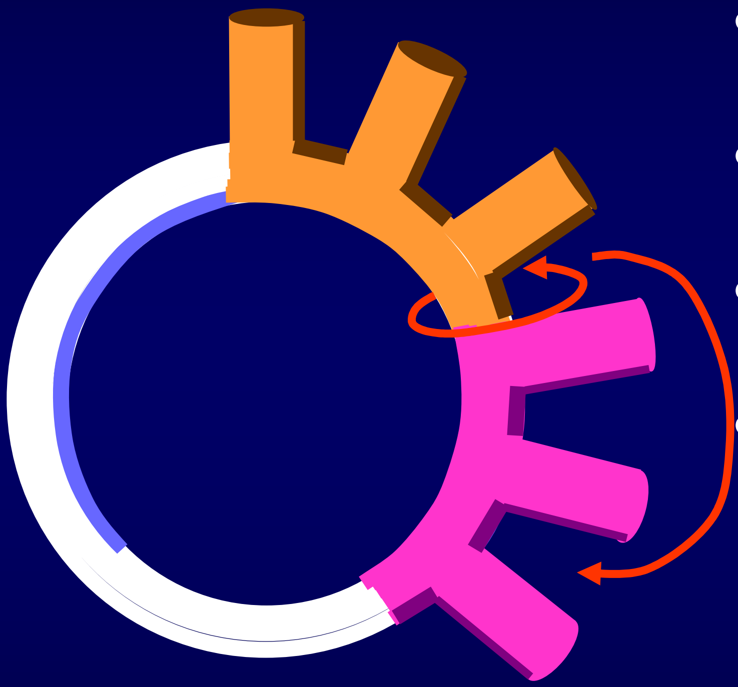
- If *X*, *Y* are basins, then *X*+*Y* is a basin
- To obtain attractors from attractors, odd number of basins must be added





Indeed:
the basin having 00…0
as attractor
is the sum of all primary basins

Type B CA60: basin composition



- take $L^{A}_{n} = 2^{\mathrm{sord}_{n}^{(2)}} - 1$ states
- rotate
- concatenate
- after n iterations, obtain a basin of type B CA, with length of $L^{B}_{n} = n\, L^{A}_{n}$
- all states with a single ‘1’ in one basin

State space construction

- take $L^A_n = 2^{\mathrm{sord}_n(2)} - 1$ states
- rotate
- add
- concatenate
- if you know how, obtain the entire state space of any CA60 (exponential time)

What is the complexity of the "instruction sheet"?

My sincere thanks