Adjacency Graph of the SNF as Source of Information

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Outline

- Introduction
- Specialized Normal Form - SNF
- Adjacency Graph of a SNF
- Extended Adjacency Graph of a SNF
- Experimental Results
- Conclusion

Introduction

- Sum-Of-Products (SOP)

$$f(a,b,c) = \bar{a}\bar{b} \vee \bar{a}\bar{c} \vee abc$$

  - well studied

- Exclusive Sum-Of-Products (ESOP)

$$f(a,b,c) = \bar{a} \oplus bc$$

  - more compact than the SOP, in general
  - less studied

c					f
0	1	1	0	0	
1	1	0	1	0	
	0	1	1	0	b
	0	0	1	1	a

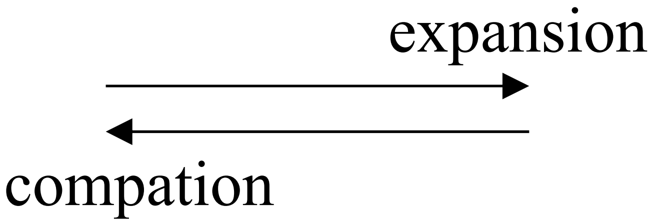
Specialized Normal Form – SNF

- algebraic property of the exclusive-or operation

$$x = \overline{x} \oplus 1 \qquad (1)$$

$$\overline{x} = 1 \oplus x \qquad (2)$$

$$1 = x \oplus \overline{x} \qquad (3)$$



- isomorphic properties of the set $\{x, \overline{x}, 1\}$

Specialized Normal Form – SNF

- Algorithm 1 Calculate Exp(f)

Require: any ESOP of a Boolean function f

Ensure: complete expansion of the Boolean function f w.r.t. all variables of its support

1: for all variables V_i of the support of f do

2: for all cubes C_j of f do

3: $\langle C_{n1}, C_{n2} \rangle \leftarrow$ **expand**(C_j , V_i)

4: replace C_j by $\langle C_{n1}, C_{n2} \rangle$

5: end for

6: end for

Specialized Normal Form – SNF

- Algorithm 1 Calculate Exp(f) / Example

$$f(a,b,c) = \overline{a}c$$

a	b	c
0	-	1

a	b	c
1	-	1
-	-	1

a	b	c
1	0	1
1	1	1
-	0	1
-	1	1

a	b	c
1	0	0
1	0	-
1	1	0
1	1	-
-	0	0
-	0	-
-	1	0
-	1	-

Specialized Normal Form – SNF

- property of the exclusive-or operation for a Boolean function f

$$f = f \oplus 0 \qquad (4)$$

$$0 = C \oplus C \qquad (5)$$

$$f = f \oplus C \oplus C \qquad (6)$$

reduction ←

- pairs of cubes can be removed from the ESOP without change of the Boolean function f

Specialized Normal Form – SNF

- Algorithm 2 Calculate R(f)

Require: any ESOP of a Boolean function f containing n cubes
Ensure: reduced ESOP of f containing no cube more than once

for $i \leftarrow 0$ to $n - 2$ do
    for $j \leftarrow i + 1$ to $n - 1$ do
        if $C_i = C_j$ then
            $C_i \leftarrow C_{n-1}$
            $C_j \leftarrow C_{n-2}$
            $n \leftarrow n - 2$
            $j \leftarrow i$
        end if
    end for
end for

Specialized Normal Form – SNF

- Definition 1 - SNF(f)

  Take any ESOP of a Boolean function f. The resulting ESOP of

$$SNF(f) = R(Exp(f)) \qquad (7)$$

  is called Specialized Normal Form (SNF) of the Boolean function.

- Exp(f) distributes the information about cubes
- R(f) removes pairs of cubes
- The SNF(f) is a unique ESOP of f.

Adjacency Graph of a SNF

- Definition 2 -Adjacency Graph $AG^{SNF(f)}(V,E)$

  The vertices V of the adjacency graph $AG^{SNF(f)}(V,E)$ correspond to the cubes of the SNF(f). Each vertex carries the ternary vector of the associated cube as label. Two vertices V of $AG^{SNF(f)}(V,E)$ are connected by an edge, if the associated labels have a distance equal to one that means they differ exactly in one position of the ternary vectors.

Adjacency Graph of a SNF

- Example

$$f(a,b,c) = \overline{a}c$$

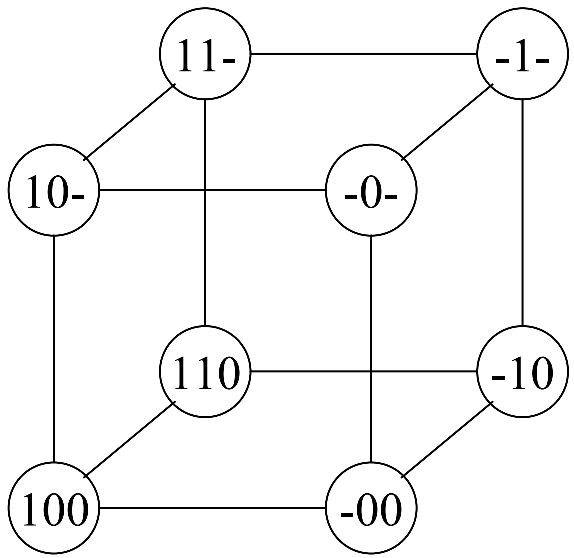
$f =$

a	b	c
0	-	1

SNF(f) =

a	b	c
1	0	0
1	0	-
1	1	0
1	1	-
-	0	0
-	0	-
-	1	0
-	1	-

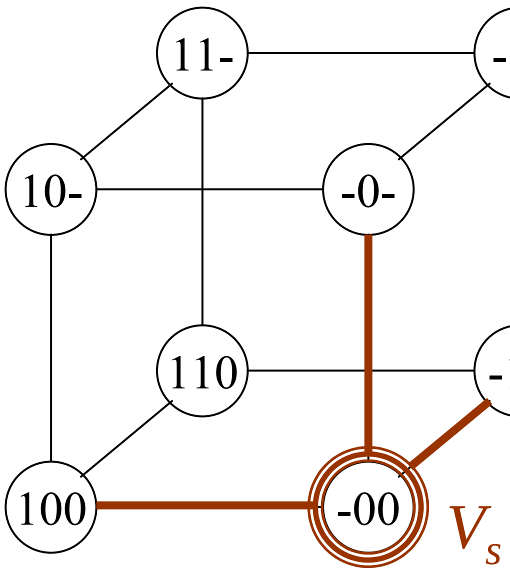
$$AG^{SNF(f)}(V,E)$$



Adjacency Graph of a SNF

- Reconstruction of the cube $f(a,b,c) = \overline{a}c$

$AG^{SNF(f)}(V,E)$

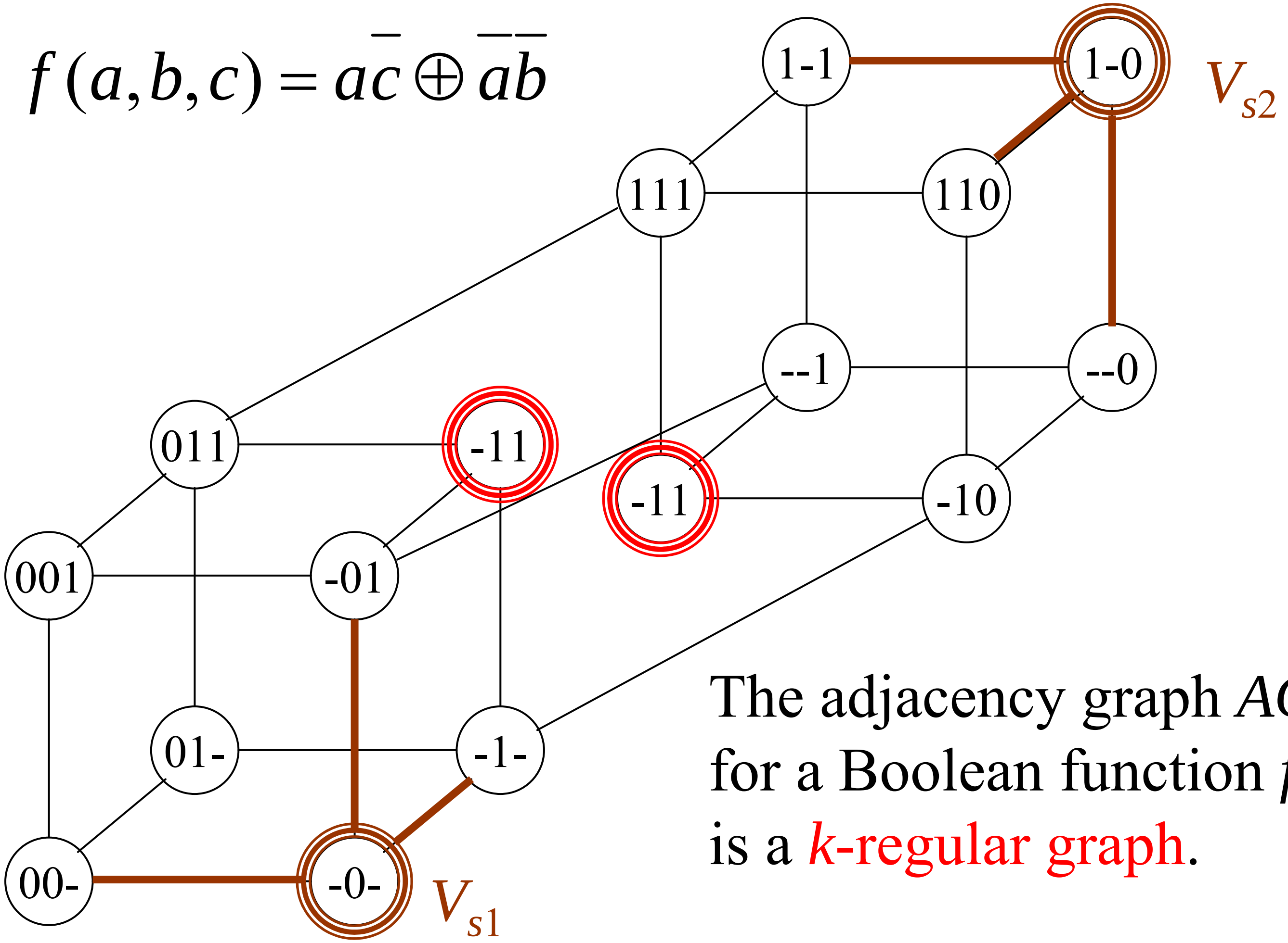


a	b	c	a	b	c		a	b	c
-	0	0	1	0	0	→	0		
-	0	0	-	1	0	→		-	
-	0	0	-	0	-	→			1
							0	-	1

HCCC(AG, V_s).
hypercube corner compaction

Adjacency Graph of a SNF

$$f(a,b,c) = a\bar{c} \oplus \bar{a}\bar{b}$$



The adjacency graph $AG^{SNF(f)}(V,E)$ for a Boolean function $f : B^k \rightarrow B$ is a *k*-regular graph.

Adjacency Graph of a SNF

- all reconstructed cubes of the adjacency graph

selected cube	neighbor cube 1	neighbor cube 2	neighbor cube 3	Created cube	associated conjunction
00-	-0-	01-	001	1-0	$a\bar{c}$
-0-	00-	-1-	-01	1-0	$a\bar{c}$
01-	-1-	00-	011	1-0	$a\bar{c}$
001	-01	011	00-	1-0	$a\bar{c}$
011	111	001	01-	--0	$\bar{c}$
-01	001	--1	-0-	110	$ab\bar{c}$
-1-	01-	-0-	-10	1-1	ac
-10	110	--0	-1-	001	$\bar{a}\bar{b}c$
--1	1-1	-01	--0	01-	$\bar{a}b$
111	011	1-1	110	-0-	b
--0	1-0	-10	--1	00-	$\bar{a}\bar{b}$
110	-10	1-0	111	00-	$\bar{a}\bar{b}$
1-1	-11	111	1-0	00-	$\bar{a}\bar{b}$
1-0	--0	110	1-1	00-	$\bar{a}\bar{b}$

Adjacency Graph of a SNF

- selection of vertex by its degree in adjacency graph

vertex

00-	**-0-**	**01-**	**001**	**011**	**-01**	**-1-**	**-10**	**--1**	**111**	**--0**	**110**	**1-1**	**1-0**
3	3	3	3	3	3	3	3	3	3	3	3	3	3

degree in adjacency graph

?

Extended Adjacency Graph of a SNF

- Definition 3

  Distance-one wrapper cubes of the SNF(*f*)

  Each cube from the same Boolean space like f that does not belong to SNF(f), but has a distance of one to at least cube of the SNF(f) is a distance-one wrapper cube of the SNF(f) .

Extended Adjacency Graph of a SNF

- Definition 4

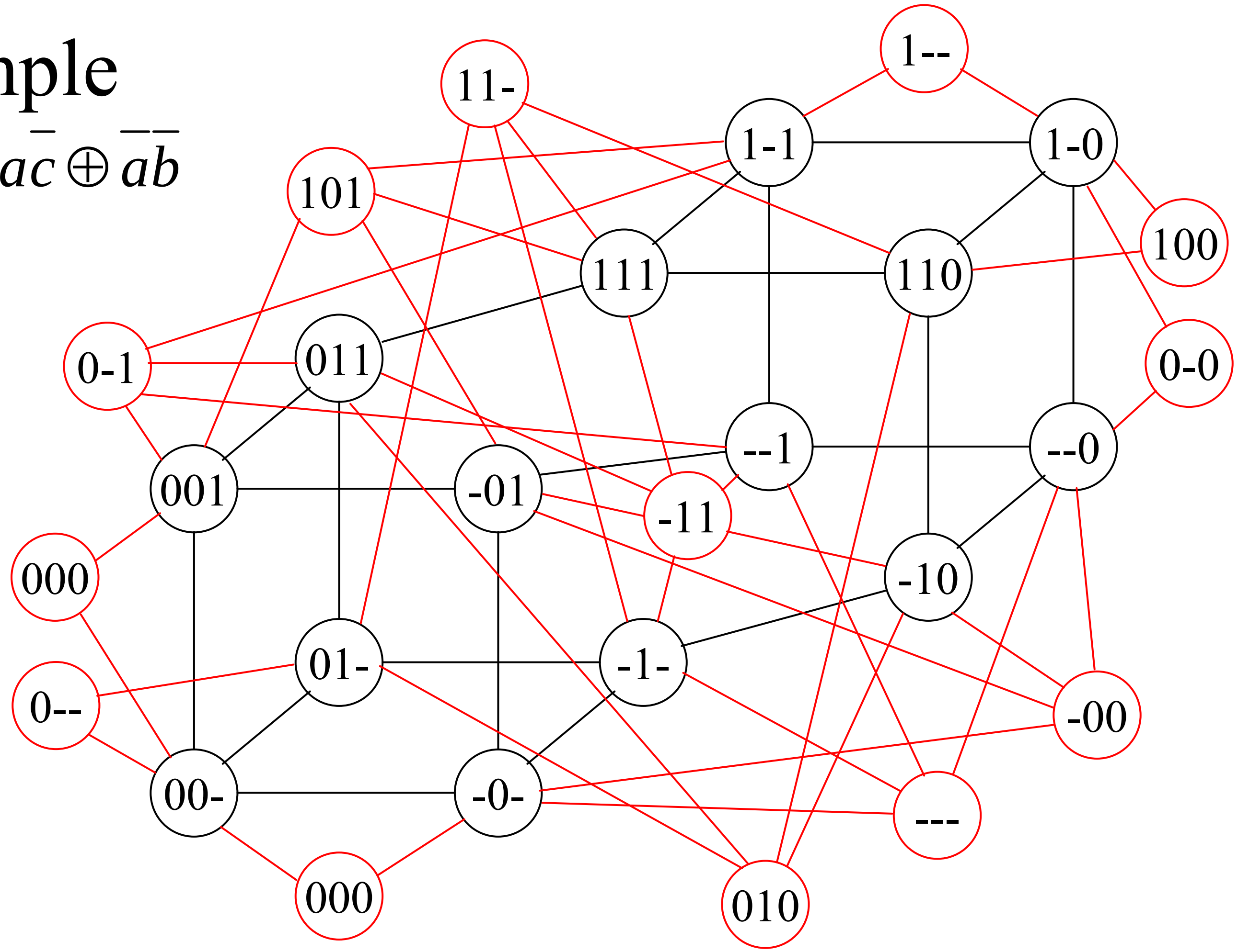
  Extended adjacency graph $\mathrm{EAG}^{\mathrm{SNF}(f)}(V,E)$ of the SNF(f)

  The extended adjacency graph of the SNF(f) consists of the adjacency graph $\mathrm{AG}^{\mathrm{SNF}(f)}(V,E)$ of the SNF(f) as core extended by vertices of all distance-one wrapper cubes of the SNF(f) and edges between these wrapper vertices and the core vertices of the SNF cubes having a distance of one. There are no edges between the wrapper vertices.

Extended Adjacency Graph of a SNF

- Example

$f(a,b,c) = a\bar{c} \oplus \bar{a}\bar{b}$



Extended Adjacency Graph of a SNF

- Algorithm 3 –

  Calculate Weights for the vertices of $EAG^{SNF(f)}(V,E)$

Require: extended adjacency graph $EAG^{SNF(f)}(V,E)$ of a Boolean function f
Ensure: weights of all vertices of $EAG^{SNF(f)}(V,E)$
1: for all wrapper vertices $V_w[i]$ of $EAG^{SNF(f)}(V,E)$ do
2: $weight(V_w[i]) \leftarrow \texttt{degree}(V_w[i])$
3: end for
4: for all core vertices $V_c[j]$ of $EAG^{SNF(f)}(V,E)$ do
5: $weight(V_c[j]) \leftarrow 0$
6: for all adjacent wrapper vertices $V_w[i]$ of $EAG^{SNF(f)}(V,E)$ do
7: $weight(V_c[j]) \leftarrow weight(V_c[j]) + weight(V_w[i])$
8: end for
9: end for

Extended Adjacency Graph of a SNF

- selection of core vertex by its weight in extended adjacency graph

vertex

00-	-0-	01-	001	011	-01	-1-	-10	--1	111	--0	110	1-1	1-0
6	10	10	10	14	14	14	14	14	14	10	10	10	6

weights of the core vertices
in extended adjacency graph

Experimental Results

- exact minimal ESOP of

$$f(a, b, c, d) = ac \oplus \bar{c} \oplus ab\bar{c} \oplus cd \oplus a\bar{b}cd$$

- $|\mathrm{SNF}(f)| = 44$
- weights

weight of core vertex	20	24	28
number of occurrence	16	20	8

- solution by HCCC(AG, V_s)

$$f(a,b,c,d) = 1 \oplus a\bar{b}\bar{c} \oplus \bar{a}c\bar{d} \oplus abcd$$

$$f(a,b,c,d) = ab \oplus cd \oplus \bar{a}\bar{c} \oplus a\bar{b}c\bar{d}$$

$$f(a,b,c,d) = \bar{c} \oplus a\bar{b} \oplus abc\bar{d} \oplus \bar{a}cd$$

$$f(a,b,c,d) = \bar{a} \oplus c\bar{d} \oplus a\bar{b}cd \oplus ab\bar{c}$$

Experimental Results

- exact minimal ESOP of all function $f : B^k \rightarrow B$, $k = 2, 3, 4$

```
# ALLBF = 16
# SNF = 0 # EMIN = 0 # BF 1
# SNF = 4 # EMIN = 1 # BF 9
# SNF = 6 # EMIN = 2 # BF 6
```

```
# ALLBF = 256
# SNF = 0  # EMIN = 0 # BF 1
# SNF = 8  # EMIN = 1 # BF 27
# SNF = 12 # EMIN = 2 # BF 54
# SNF = 14 # EMIN = 2 # BF 108
# SNF = 16 # EMIN = 3 # BF 54
# SNF = 18 # EMIN = 3 # BF 12
```

```
# ALLBF = 65536
# SNF = 0  # EMIN = 0 # BF 1
# SNF = 16 # EMIN = 1 # BF 81
# SNF = 24 # EMIN = 2 # BF 324
# SNF = 28 # EMIN = 2 # BF 1296
# SNF = 30 # EMIN = 2 # BF 648
# SNF = 32 # EMIN = 3 # BF 648
# SNF = 34 # EMIN = 3 # BF 3888
# SNF = 36 # EMIN = 3 # BF 6624
# SNF = 36 # EMIN = 4 # BF 108
# SNF = 38 # EMIN = 3 # BF 7776
# SNF = 40 # EMIN = 3 # BF 2592
# SNF = 40 # EMIN = 4 # BF 6642
# SNF = 42 # EMIN = 3 # BF 216
# SNF = 42 # EMIN = 4 # BF 14256
# SNF = 44 # EMIN = 4 # BF 12636
# SNF = 46 # EMIN = 4 # BF 3888
# SNF = 46 # EMIN = 5 # BF 1296
# SNF = 48 # EMIN = 5 # BF 1944
# SNF = 50 # EMIN = 5 # BF 648
# SNF = 54 # EMIN = 6 # BF 24
```

Conclusion

- new approach to reconstruct an exact minimal ESOP from SFN(f)
- HCCC function utilizes the k-regularity of the adjacency graph
- minimal weights of core vertices in the $EAG^{SNF(f)}(V,E)$ indicate cubes of exact minimal ESOP
- defect of the old method is removed by the new approach
- only the weight of the SNF cubes and not all 3^k cubes must be evaluated
- for a small number of Boolean functions all weights of the core vertices are equal
- a slightly changed HCCC function is necessary in this case
- new classification of Boolean functions based on both, the number of cubes in their SNFs and the number of cubes in their exact minimal ESOPs
- this classification may be a root for further theoretical results to extends the knowledge about Boolean functions