



# Maitra Cascade Minimization

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# Basic Definitions(1)

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- Complex Maitra Term [1]
  - Constant 0(1) Boolean Function
  - Literal
  - $M_{i+1} = G(a, M_i)$ 
    - $M_i \rightarrow$  Maitra Term
    - $a \rightarrow$  literal
    - $G \rightarrow$  Arbitrary 2 variable switching function ( $\sim$  Maitra Cell)
- Reversible Wave Cascade expression (Maitra expression) [1]
  - $$Q = \sum_{i=1}^m M_i$$

# Basic Definitions(2)

- Minimal (Exact) expression:
  - Least number of complex terms (weight)
- Cascade realizable function(f)  
 $\Leftrightarrow w(f)=1$
- Restricted Maitra Cascade
  - G implements a complete set of 6 functions (Table 1)

Table 1: Cell index set

Cell index(r)	$F_r(x, y)$
1	$x + y$
2	$\bar{x} + y$
3	$\bar{x}y$
4	$xy$
5	$x \oplus y$
6	$y$



# Boolean Decompositions

- Shannon, Negative Davio, Positive Davio decompositions
- New Decompositions

$$f(X) = \bar{x}f_0 \oplus xf_1$$

$$f(X) = \bar{x}f_2 \oplus f_0$$

$$f(X) = xf_2 \oplus f_1$$

$$f(X) = (x + f_2) \oplus (x \oplus f_1)$$

$$f(X) = (x + f_0) \oplus (x \bar{f}_1)$$

$$f(X) = (x \bar{f}_2) \oplus (x \oplus f_0)$$

$$f(X) = (x + \bar{f}_0) \oplus (\bar{x} \oplus \bar{f}_1)$$

$$f(X) = (\bar{x} + \bar{f}_2) \oplus \bar{f}_0$$

$$f(X) = (x + \bar{f}_2) \oplus \bar{f}_1$$

$$f(X) = (\bar{x} \bar{f}_2) \oplus (x \oplus \bar{f}_1)$$

$$f(X) = (\bar{x} \bar{f}_0) \oplus (\bar{x} + f_1)$$

$$f(X) = (\bar{x} + f_2) \oplus (x \oplus \bar{f}_0)$$

# Cascade Merging

## Lemma 1:

Relation:  $F_{r_1}(x, y_1) \oplus F_{r_2}(x, y_2) = F_r(x, y_1 \oplus y_2)$ ,  $y_1 \neq y_2$ ,  $y_1 \neq \overline{y_2}$

holds iff:

$$(r_1, r_2, r) = (1,1,3), (1,3,1), (2,2,4), (2,4,2), (3,3,3), (4,4,4), (5,5,6), (5,6,5), (6,6,6)$$

## Example:

$$[(x_1 + \overline{x_2})\overline{x_3}x_4] \oplus [((x_1 + \overline{x_2}) \oplus x_3) + \overline{x_4}] = \{[(x_1 + \overline{x_2})\overline{x_3}] \oplus [((x_1 + \overline{x_2}) \oplus x_3)]\} + \overline{x_4}$$

$\Leftrightarrow$

$$1234 \oplus 1252 = (123 \oplus 125)2$$



# Maitra Cell Classes

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- Cell has non-constant inputs:

Cell Class	Cell Index 1	Cell Index 2 (representative)
1	1	3
2	2	4
3	5	6

- Cell with a constant input (cascaded 0)

Cell Class	Cell Index 1 (representative)	Cell Index 2
1	1	2
2	6	



# Relative Complex Terms (1)

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- Two complex terms

$$P_1 = R_1 R_2 \dots R_n, P_2 = Q_1 Q_2 \dots Q_n$$

with  $R_i, Q_i$  maitra cells are relatives if  $R_i$  and  $Q_i, i \leq n$  belong to the same maitra cell class.

- The following complex terms are relatives:

$$P, \overline{P}$$

$$P, P \oplus x_n$$

$$P, P \oplus \overline{x_n}$$



# Relative Complex Terms (2)

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- Maitra Generator Class.
    - Maitra expressions with relative corresponding terms.
  - Equivalent maitra expressions
    - Represent the same switching function.
    - Belong to the same generator class.
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## Example

Expressions:  $Q_1 = (1234) \oplus (6661)$ ,  $Q_2 = (2412) \oplus (6662)$   
are equivalents since they both represent the same switching function and furthermore terms: 1234 & 2412 are relatives and terms 6661 & 6662 are relatives.

# Relative Complex Terms (3)

CIL: Number of cells with one constant input 0.

Lemma 2:

$P_1, P_2$  relative complex terms  $\rightarrow P_1 \oplus P_2$  is a complex term.

## Example

$(123455) \oplus (241466) = 666155$	$(123455)$ and $(241466)$ are relatives
$(123) = \overline{(241)}$	$(123)$ and $(241)$ are complements
$(1234) \oplus (6661) = (2414)$	$(6661) = x_4$ and $(1234) \oplus x_4 = (x_1 + \overline{x_2}) \overline{x_3} x_4 \oplus x_4 =$ $(\overline{(x_1 x_2)} + x_3) x_4 = (2414)$

# Relative Complex Terms (4)

Lemma 3: If  $P, P_1, M, M_1$  are complex terms and:

- $M \oplus P = P_1$
- $P, P_1$  relatives
- $M, M_1$  have
  - $CIL(M_1) \geq CIL(M) > 0$
  - cells of same class from  $CIL(M_1)+2$  until the last cell

Then:  $M_1 \oplus P = P_2$  and  $P, P_2$  are relative complex terms

## Example

$$(1243) \oplus (6123) = (2243)$$

&

$$(1243) \oplus (6611) = (2441)$$

(6123) & (6611) have  $CIL=1$  &  $2$  respectively  
and (2243) & (2441) are relatives



# Relative Complex Terms (5)

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Lemma 4: If  $P_1, P_2$  are complex terms and:

- $0 < CIL(P_1) < CIL(P_2)$
- cells of the same class from  $CIL(P_2)+2$  until the last cell

Then  $P_1 \oplus P_2$  is a complex term and is relative  $P_1$ .

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Example

$$(666613) \oplus (661433) = (661413)$$

# Complex Term Splitting

Lemma 5 (Term splitting):

$$P_1 = p_1 p_2 \dots p_i \dots p_n =$$

$$Q_1 \oplus Q_2 =$$

$$p_1^1 p_2^1 \dots q_1 \dots p_n^1 \oplus p_1^2 p_2^2 \dots q_2 \dots p_n^2$$

- $P_1, Q_1, Q_2$  complex terms
- $p_j, p_j^1, p_j^2, j \leq n, j \neq i$  cells of the same class
- $p_i, q_1, q_2$  cells of different cell class.

Example

$$(1234) = (1264) \oplus (1244)$$

$$(1232) = (2411) \oplus (1235)$$

Lemma 6:

- $P_1, P_2$  relative complex terms
- $P_1, P_2$  are split at position  $i$ :  

$$P_1 = P_{11} \oplus P_{12}, P_2 = P_{21} \oplus P_{22}$$
- $P_{11}, P_{21}$  and  $P_{21}, P_{22}$  are relatives.

Example

$$(1233) = (1234) \oplus (1236)$$

$$(1411) = (1414) \oplus (1415)$$

# Minimization Theorems (1)

- Theorem 1: Each minimal expression of  $f$  can be expressed as:

- $f = F_p(x_1, y)$

- $f = F_p(x_1, y) \oplus F_q(x_1, z)$

- $f = F_p(x_1, y) \oplus F_q(x_1, z) \oplus F_r(x_1, g)$

• Constant subfunction

•  $(p,q)=(3,4), (3,6), (4,6)$   
and  $y, z$  subfunctions

•  $(p,q,r)=(3,4,6)$   
 $f_0, f_1, f_2 = y \oplus g,$   
 $z \oplus g, y \oplus z$

+ Equiv.  
forms

Proof

XOR-sum minimal form of  $f$ :

$$f(x_1, \dots, x_n) = F_{r_1}(x_1, y_1) \oplus \dots \oplus F_{r_n}(x_1, y_n)$$

*Comparison with Lemma 1.*



# Minimization Theorems (2)

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## Theorem 2:

At least one minimal expression of a switching function

$f(x_1, x_2, \dots, x_n)$  with less than 6 variables can be obtained from the minimal expressions of its subfunctions.

## Proof:

3 cases (Theorem 1):

- $Expr(f) = F_p(x, y)$ 
  - Theorem 1  $\rightarrow$  Constant subfunction.
  - $Expr(f)$  produced from the minimal expression of non constant subfunctions

# Minimization Theorems (3)

- $Expr(f) = F_p(x, y) \oplus F_q(x, z)$ ,  $y, z$  subfunctions.
  - $w(f) = w(y) + w(z)$
  - $Expr(f)$  produced by the minimal expressions of subfunctions.
- $Expr(f) = F_p(x, y) \oplus F_q(x, z) \oplus F_r(x, g)$ ,  $f_0, f_1, f_2 = y \oplus g, z \oplus g, y \oplus z$ 
  - $w(y) \leq w(z) \leq w(g) \rightarrow w(f) \leq 5 \rightarrow w(y) = 1$ 
    - $w(f) = 3 \rightarrow w(y) = w(z) = w(g) = 1 \rightarrow w(f_i) = w(f_j) = 2 \rightarrow 1$  common term
    - $w(f) = w(y) + w(z) + w(g) = m = 4, 5 \rightarrow w(y) = 1 \rightarrow m \leq w(f_0) + w(f_2) \leq 2 * w(y) + w(g) + w(z) = m + 1$ 
      - $w(f_0) + w(f_2) = m$ . Proof follows that of previous case.
      - $w(f_0) + w(f_2) = m + 1 \rightarrow f_0 = y \oplus g, f_2 = y \oplus z$  (minimal exprs of these subfunctions for this particular case), At least one minimal expr of  $f_0$  and  $f_2$  with one common term ( $y$ ) to be merged.



# Minimization Theorems (4)

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## Theorem 3:

Let  $Q_1$  be a minimal expression of  $f$ , produced by the minimal expressions  $F_{i1}, F_{j1}$  of  $f_i, f_j$  respectively. An equivalent to  $Q_1$  expression  $Q_2$  of  $f$  can be obtained from two other minimal expressions  $F_{i2}, F_{j2}$  of  $f_i, f_j$  which are equivalents to  $F_{i1}, F_{j1}$ .

# Minimization Theorems (5)

## Proof:

$$Q_1 \leftarrow F_{i1} = P_{i11} \oplus P_{i12} \oplus \dots \oplus P_{i1q}, F_{j1} = P_{j11} \oplus P_{j12} \oplus \dots \oplus P_{j1r}$$

$$Q_2 \leftarrow F_{i2} = P_{i21} \oplus P_{i22} \oplus \dots \oplus P_{i2q}, F_{j2} = P_{j21} \oplus P_{j22} \oplus \dots \oplus P_{j2r}$$

$$Q_1 = (P_{i11} \oplus \dots \oplus P_{i1q})k'_1 \oplus (P_{j11} \oplus \dots \oplus P_{j1r})k'_2 = P_{i11}k'_3 \oplus (P_{i12} \oplus \dots \oplus P_{i1q})k'_1 \oplus (P_{j12} \oplus \dots \oplus P_{j1r})k'_2$$

- If  $w(f) = w(f_i)$  ( $i=0,1,2$ ) then the proof is trivial.
- If  $w(f) = w(f_i) + w(f_j)$  then the proof is trivial.
- If  $w(f) = w(f_i) + w(f_j) - 1$ . Common term:  $P_{i11} = P_{j11}$   
 $\rightarrow P_{i11}, P_{j11}, P_{i21}, P_{j21}$  relatives.

□ It holds:

$$F_{i1} \oplus F_{i2} = 0 \Leftrightarrow (P_{i12} \oplus \dots \oplus P_{i1q}) \oplus (P_{i22} \oplus \dots \oplus P_{i2q}) = P_{i11} \oplus P_{i21} = M_1$$

$$F_{j1} \oplus F_{j2} = 0 \Leftrightarrow (P_{j12} \oplus \dots \oplus P_{j1r}) \oplus (P_{j22} \oplus \dots \oplus P_{j2r}) = P_{j11} \oplus P_{j21} = M_2$$

# Minimization Theorems (6)

Furthermore:  $M_1 \oplus M_2 = P_{i21} \oplus P_{j21} = M$  and  $M_1, M_2$  will have cells of the same cell class from position  $\text{MAX}(\text{CIL}(M_1), \text{CIL}(M_2)) + 2$  until the last cell (Lemma 2). Without loss of generality:  $\text{CIL}(M_2) \leq \text{CIL}(M_1)$ .

- $\text{CIL}(M_2) < \text{CIL}(M_1) \rightarrow M$  is relative to  $M_2$  (Lemma 4).

$$(P_{j12} \oplus \dots \oplus P_{j1r}) = M_2 \oplus (P_{j22} \oplus \dots \oplus P_{j2r}) \quad \text{and}$$

$$Q_2 = (P_{i21}k_1 \oplus P_{i22}k_2) \oplus (P_{i22} \oplus \dots \oplus P_{i2r})k_1 \oplus (P_{j22} \oplus \dots \oplus P_{j2r})k_2 \Leftrightarrow$$

$$Q_2 = (P_{i21}k_3 \oplus P_{i21}k_2 \oplus P_{i22}k_2) \oplus (P_{i22} \oplus \dots \oplus P_{i2r})k_1 \oplus (P_{j22} \oplus \dots \oplus P_{j2r})k_2 \Leftrightarrow$$

$$Q_2 = P_{i21}k_3 \oplus (P_{i22} \oplus \dots \oplus P_{i2r})k_1 \oplus (M \oplus P_{j22} \oplus \dots \oplus P_{j2r})k_2$$

- $r \leq 3 \rightarrow M$  merges with at most 2 terms

- $\text{CIL}(M_2) = \text{CIL}(M_1) \rightarrow M$  will have cells of the same class (with  $M_1$  &  $M_2$ ) from position  $x$  ( $> \text{CIL}(M_2) + 2$ ) until the last cell, while the rest of them will represent a  $\oplus_x, \oplus_x$  literal. The proof follows that of previous case (Lemma 3).

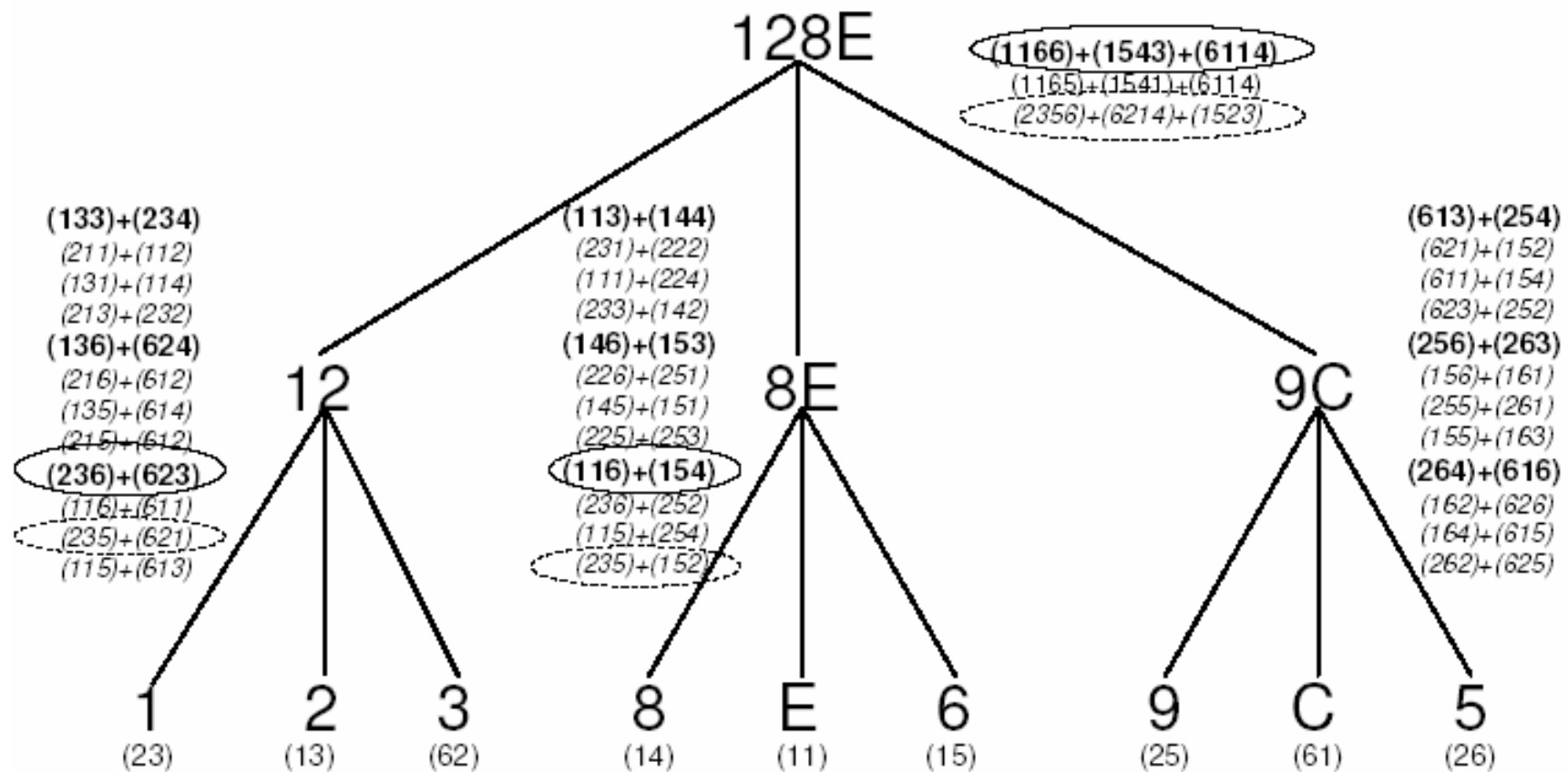


# Heuristic Algorithm outline

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- Input : a switching function in minterm formulation
- Decomposition (recursively) using ETDDs (Shannon and Davio expansions).
- Composition (recursively): Produce expressions for  $f$  from the minimal expressions of its subfunctions.
  - If complex terms with the same generator are found between two such expressions ( $F_i, F_j$ ):
    - Merge them to produce a by-product complex term.
    - Try to merge that by-product with the rest of the terms in  $F_i, F_j$ .
  - If this by-product is merged, then the weight of the function is reduced by 1.
- At the end keep those expressions with the least number of complex terms.

# Example



# Experimental results

Lee's Algorithm: *G.Lee, R.Drechsler: ETDD-Based Synthesis of term-based FPGAs for incompletely specified boolean functions, ASP-DAC 1998*

Table 3: Single-output MCNC Functions

Function	Lee's Algo	Minict	EMin1
	#	#	#
t481	10	12	10
9sym	-	45	32
xor5	-	1	1

Minict: *N. Song, M. Perkowski: Minimization of exclusive sums of multi-valued complex terms for logic cell arrays, ISMVL 98, p 32*

Table 4: Multi-Output MCNC Functions

Function	EMin1	Function	EMin1	Function	EMin1	Function	EMin1
	#		#		#		#
rd53:1	4	misex1:1	2	5xp1:1	5	inc:1	5
rd53:2	1	misex1:2	4	5xp1:2	9	inc:2	6
rd53:3	4	misex1:3	5	5xp1:3	7	inc:3	6
rd73:1	6	misex1:4	4	5xp1:4	6	inc:4	9
rd73:2	1	misex1:5	2	5xp1:5	3	inc:5	5
rd73:3	12	misex1:6	4	5xp1:6	2	inc:6	2
rd84:1	7	misex1:7	4	5xp1:7	1	inc:7	2
rd84:2	1	sao2:1	10	5xp1:8	1	inc:8	4
rd84:3	1	sao2:2	12	5xp1:9	1	inc:9	1
rd84:4	21	sao2:3	12	5xp1:10	3	con1:1	5
		sao2:4	11			con1:2	4



# Conclusions and Future work

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## ■ Contribution

- New boolean decompositions are presented.
- An algorithm has been described for producing minimal reversible wave cascades for switching functions up to 5 variables and near minimal for functions with more variables, without the need to calculate equivalent expressions.

## ■ Future work

- Multi-output switching functions.
- Minimal solution for functions of up to 6 variables.



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