Derivative Operations for Lattices of Boolean Functions

Bernd Steinbach
Institute of Computer Science
Freiberg University of Mining and Technology
Freiberg, Germany
Email: steinb@informatik.tu-freiberg.de

Christian Posthoff
Department of Computing and Information Technology
The University of the West Indies
Trinidad & Tobago
Email: christian@posthoff.de

***Abstract*—This paper explores derivative operations of the Boolean differential calculus for lattices of Boolean functions. Such operations are needed to design circuits with short delay and low power consumption [3] as well as to calculate minimal complete sets of fitting test patterns [4].**

It will be shown that each derivative operation of a lattice of Boolean functions creates again a lattice of Boolean functions. The created lattice can be the same lattice as the given one, but in most cases the created lattice of Boolean functions is *simpler* than the given lattice.

There is a direct mapping of an incompletely specified Boolean function to a lattice of Boolean functions. We will show that such lattices of Boolean functions are only a *subclass* of all lattices of Boolean functions. A unique general specification of a lattice of Boolean functions will be given.

***Index Terms*—Boolean function; lattice; Boolean Differential Calculus; derivative operation; incompletely specified Boolean function; independency matrix; independency function; rank.**

I. Introduction

A Boolean function $f(x_1, \ldots, x_n)$ of n variables is a unique mapping from B^n into B, where $B = \{0, 1\}$ [2]. We will use the Boolean space B^n in this paper as fixed common space. It is well known that a completely specified Boolean function $f(x_1, \ldots, x_n) = f(\mathbf{x})$ divides the 2^n patterns $\mathbf{x}$ of the Boolean space B^n in two disjoint sets:

$$\begin{aligned} &\mathbf{x} \in ON-set \text{ (completely specified)} \\ &\quad \Leftrightarrow f_{csq}(x_1, \ldots, x_n) = 1 \\ &\quad \Leftrightarrow f(x_1, \ldots, x_n) = 1 \ , \end{aligned} \tag{1}$$

$$\begin{aligned} &\mathbf{x} \in OFF-set \text{ (completely specified)} \\ &\quad \Leftrightarrow f_{csr}(x_1, \ldots, x_n) = 1 \\ &\quad \Leftrightarrow f(x_1, \ldots, x_n) = 0 \ . \end{aligned} \tag{2}$$

In some cases not all 2^n function values must be specified to the fixed values 0 or 1. Such an incompletely specified Boolean function divides the 2^n patterns $\mathbf{x}$ of the Boolean space B^n into three disjoint sets:

$$\begin{aligned} &\mathbf{x} \in don't-care-set \\ &\quad \Leftrightarrow f_{\varphi}(x_1, \ldots, x_n) = 1 \\ &\quad \Leftrightarrow \text{it is allowed to choose the function value of} \\ &\qquad f(\mathbf{x}) \text{ without any restrictions,} \end{aligned} \tag{3}$$

$$\begin{aligned} &\mathbf{x} \in ON-set \\ &\quad \Leftrightarrow f_q(x_1, \ldots, x_n) = 1 \\ &\quad \Leftrightarrow (f_{\varphi}(x_1, \ldots, x_n) = 0) \wedge (f(x_1, \ldots, x_n) = 1) \ , \end{aligned} \tag{4}$$

$$\begin{aligned} &\mathbf{x} \in OFF-set \\ &\quad \Leftrightarrow f_r(x_1, \ldots, x_n) = 1 \\ &\quad \Leftrightarrow (f_{\varphi}(x_1, \ldots, x_n) = 0) \wedge (f(x_1, \ldots, x_n) = 0) \ . \end{aligned} \tag{5}$$

These mark functions $f_q(\mathbf{x})$, $f_r(\mathbf{x})$ and $f_{\varphi}(\mathbf{x})$ cover the whole Boolean space,

$$f_q(\mathbf{x}) \vee f_r(\mathbf{x}) \vee f_{\varphi}(\mathbf{x}) = 1 \tag{6}$$

for all vectors $\mathbf{x}$, and they are also mutually disjoint:

$$f_q(\mathbf{x}) \wedge f_r(\mathbf{x}) = 0 \ , \tag{7}$$
$$f_q(\mathbf{x}) \wedge f_{\varphi}(\mathbf{x}) = 0 \ , \tag{8}$$
$$f_r(\mathbf{x}) \wedge f_{\varphi}(\mathbf{x}) = 0 \ . \tag{9}$$

Incompletely specified Boolean functions are very useful in circuit design. Due to (3) the function to be implemented by the circuit can be chosen out of a set of Boolean functions, not only a single function defined by (1) and (2) can be used. Even if a completely specified function is given, the design method of bi-decomposition [1], [5], [6] for multilevel circuit structures takes strong advantage from the created function set of the decomposed subfunctions. The used function sets correspond directly to incompletely specified functions as introduced above, and all Boolean functions $f(x_1, \ldots, x_n) = f(\mathbf{x})$ which satisfy the inequality (10) belong to this set:

$$f_q(\mathbf{x}) \leq f(\mathbf{x}) \leq \overline{f_r(\mathbf{x})} \tag{10}$$

The inequality (10) contains two separate inequalities which can be transformed into restrictive Boolean equations:

$$f_q(\mathbf{x}) \leq f(\mathbf{x}) \quad \Leftrightarrow \quad f_q(\mathbf{x}) \wedge \overline{f(\mathbf{x})} = 0 \ , \tag{11}$$
$$f(\mathbf{x}) \leq \overline{f_r(\mathbf{x})} \quad \Leftrightarrow \quad f(\mathbf{x}) \wedge f_r(\mathbf{x}) = 0 \ . \tag{12}$$

All Boolean functions $f(x_1, \ldots, x_n) = f(\mathbf{x})$ which satisfy the inequality (10) define the basic set of a lattice. This lattice is closed regarding the operations *AND* ($\wedge$) and *OR* ($\vee$). That means: if

$$f_q(\mathbf{x}) \leq f_1(\mathbf{x}) \leq \overline{f_r(\mathbf{x})} \tag{13}$$

and

$$f_q(\mathbf{x}) \le f_2(\mathbf{x}) \le \overline{f_r(\mathbf{x})} \tag{14}$$

then it holds that

$$f_q(\mathbf{x}) \le f_1(\mathbf{x}) \wedge f_2(\mathbf{x}) \le \overline{f_r(\mathbf{x})} \tag{15}$$

and

$$f_q(\mathbf{x}) \le f_1(\mathbf{x}) \vee f_2(\mathbf{x}) \le \overline{f_r(\mathbf{x})} \ . \tag{16}$$

The infimum (the 0-element) of this lattice is $f_q(\mathbf{x})$ and the supremum (the 1-element) of this lattice is $\overline{f_r(\mathbf{x})}$. Furthermore, this lattice is complementary which means that for each function $f(\mathbf{x})$ which satisfies the inequality (10) there is a complement function ${}^L\overline{f(\mathbf{x})}$

$${}^L\overline{f(\mathbf{x})} = \overline{f(\mathbf{x})} \wedge \overline{f_r(\mathbf{x})} \vee f_q(\mathbf{x}) \tag{17}$$

such that

$${}^L\overline{f(\mathbf{x})} \wedge f(\mathbf{x}) = f_q(\mathbf{x}) \ , \tag{18}$$

$${}^L\overline{f(\mathbf{x})} \vee f(\mathbf{x}) = \overline{f_r(\mathbf{x})} \ . \tag{19}$$

The validity of (18) can easily be shown by substitution of (17) into (18) and the use of (11):

$$\begin{aligned}
{}^L\overline{f(\mathbf{x})} \wedge f(\mathbf{x}) &= f_q(\mathbf{x}) \\
(\overline{f(\mathbf{x})} \wedge \overline{f_r(\mathbf{x})} \vee f_q(\mathbf{x})) \wedge f(\mathbf{x}) &= f_q(\mathbf{x}) \\
f(\mathbf{x}) \wedge \overline{f(\mathbf{x})} \wedge \overline{f_r(\mathbf{x})} \vee f(\mathbf{x}) \wedge f_q(\mathbf{x}) &= f_q(\mathbf{x}) \\
f(\mathbf{x}) \wedge f_q(\mathbf{x}) \vee f_q(\mathbf{x}) \wedge \overline{f(\mathbf{x})} &= f_q(\mathbf{x}) \\
f_q(\mathbf{x}) \wedge (f(\mathbf{x}) \vee \overline{f(\mathbf{x})}) &= f_q(\mathbf{x}) \\
f_q(\mathbf{x}) &= f_q(\mathbf{x}) \ .
\end{aligned} \tag{20}$$

Similarly, the validity of (19) can be shown by substituting (17) into (19) and the use of both (11) and (12):

$$\begin{aligned}
{}^L\overline{f(\mathbf{x})} \vee f(\mathbf{x}) &= \overline{f_r(\mathbf{x})} \\
(\overline{f(\mathbf{x})} \wedge \overline{f_r(\mathbf{x})} \vee f_q(\mathbf{x})) \vee f(\mathbf{x}) &= \overline{f_r(\mathbf{x})} \\
\overline{f(\mathbf{x})} \wedge \overline{f_r(\mathbf{x})} \vee f_q(\mathbf{x}) \vee f(\mathbf{x}) &= \overline{f_r(\mathbf{x})} \\
\overline{f(\mathbf{x})} \wedge \overline{f_r(\mathbf{x})} \vee f(\mathbf{x}) &= \overline{f_r(\mathbf{x})} \\
\overline{f_r(\mathbf{x})} \vee f(\mathbf{x}) &= \overline{f_r(\mathbf{x})} \\
\overline{f_r(\mathbf{x})} \vee f(\mathbf{x}) \wedge (f_r(\mathbf{x}) \vee \overline{f_r(\mathbf{x})}) &= \overline{f_r(\mathbf{x})} \\
\overline{f_r(\mathbf{x})} \vee f(\mathbf{x}) \wedge f_r(\mathbf{x}) \vee f(\mathbf{x}) \wedge \overline{f_r(\mathbf{x})} &= \overline{f_r(\mathbf{x})} \\
\overline{f_r(\mathbf{x})} \vee f(\mathbf{x}) \wedge \overline{f_r(\mathbf{x})} &= \overline{f_r(\mathbf{x})} \\
\overline{f_r(\mathbf{x})} &= \overline{f_r(\mathbf{x})} \ .
\end{aligned} \tag{21}$$

The benefit of a lattice of Boolean functions in circuit design is the possibility to select a simple function from the set of functions that satisfy (10). The drawback of a lattice of Boolean functions is the need to execute the required operations for all functions of the lattice. Besides the elementary Boolean operations especially derivative operations of the Boolean differential calculus [2], [7], [8] must be used.

We assume that a certain derivative operation must be calculated for all Boolean functions of a lattice. It can be expected that the calculation of this derivative operation for all functions of the given function set creates a new set of Boolean functions. It is the aim of this paper to answer the following questions.

1) Does the created set of Boolean functions hold the axioms of a lattice of Boolean functions?
2) If *YES*, how the created lattice can be described in a general way?
3) How many functions belong to the function set created by the executed derivative operation?
4) Are the functions of the created set simpler or more difficult in comparison to functions of the given lattice?

TABLE I
TAXONOMIE OF DERIVATIVE OPERATIONS

basic operation	number of involved variables: 1	> 1, execution of changes: sequentially	> 1, execution of changes: simultaneously
$\oplus$	$\frac{\partial f(x_i,\mathbf{x_1})}{\partial x_i}$	$\frac{\partial^m f(\mathbf{x}_0,\mathbf{x_1})}{\partial x_{01}\ldots\partial x_{0m}}$	$\frac{\partial f(\mathbf{x}_0,\mathbf{x_1})}{\partial \mathbf{x}_0}$
$\wedge$	$\min_{x_i} f(x_i,\mathbf{x_1})$	$\min^m_{\mathbf{x}_0} f(\mathbf{x}_0,\mathbf{x}_1)$	$\min_{\mathbf{x}_0} f(\mathbf{x}_0,\mathbf{x}_1)$
$\vee$	$\max_{x_i} f(x_i,\mathbf{x_1})$	$\max^m_{\mathbf{x}_0} f(\mathbf{x}_0,\mathbf{x}_1)$	$\max_{\mathbf{x}_0} f(\mathbf{x}_0,\mathbf{x}_1)$

The rest of the paper is organized as follows. Section II gives Definitions which make the paper self-contained. Dependencies of derivative operations of their direction of change are explored in detail in Section III. A simple description of a so far unused type of lattices of Boolean functions is introduced in Section IV and used for derivative operations in Section V.

II. DERIVATIVE OPERATION OF BOOLEAN FUNCTIONS

A large number of applications of the Boolean Differential Calculus (BDC) has been published in [8]. A short introduction into the BDC in connection with some selected applications is given in [7]. One chapter of [2] gives an extended introduction into the BDC. Here we confine ourselves to the derivative operations.

The derivative operations of the BDC can be classified on the basis of a very simple taxonomy. The first question to distinguish these operations asks for the basic operation which can be the EXOR ($\oplus$) for the derivative, the AND ($\wedge$) for the minimum or the OR ($\vee$) for the maximum. The second question to distinguish these operations asks for the number of involved variables which can be one single variable for simple derivative operations or more than one variable. In case of several involved variables, the last question asks whether a sequence of changes of all these variables is studied or all selected variables change their values simultaneously. In case of a sequence of changes we get m-fold derivative operations and in case of the simultaneous change of all selected variables the vectorial derivative operations. Table I shows this taxonomy.

The simple derivative operations verify the change behavior of the given Boolean function depending on the change of of the value a single variable x_i.

Definition 1: Let $f(\mathbf{x}) = f(x_i, \mathbf{x}_1)$ be a Boolean function of n variables. Then

$$\frac{\partial f(\mathbf{x})}{\partial x_i} = f(x_i, \mathbf{x}_1) \oplus f(\overline{x}_i, \mathbf{x}_1) \tag{22}$$

is the (simple) derivative of this function with regard to x_i,

$$\min_{x_i} f(\mathbf{x}) = f(x_i, \mathbf{x}_1) \wedge f(\overline{x}_i, \mathbf{x}_1) \tag{23}$$

the (simple) minimum of this function with regard to x_i, and

$$\max_{x_i} f(\mathbf{x}) = f(x_i, \mathbf{x}_1) \vee f(\overline{x}_i, \mathbf{x}_1) \tag{24}$$

the (simple) maximum of this function with regard to x_i.

The m-fold derivative operations of the given Boolean function can be calculated as a sequence of simple derivative operations. The result of these operations does not depend on the order of the variables in the sequence of the calculations because all three basic operations are commutative. Each function value of an m-fold derivative operation carries the information of function values of the given function in a certain subspace.

Definition 2: Let $f(\mathbf{x}) = f(\mathbf{x}_0, \mathbf{x}_1)$ be a Boolean function of n variables. Then

$$\frac{\partial^m f(\mathbf{x}_0, \mathbf{x}_1)}{\partial x_{01} \partial x_{02} \ldots \partial x_{0m}} = \frac{\partial}{\partial x_{0m}}(\ldots(\frac{\partial}{\partial x_{02}}(\frac{\partial f(\mathbf{x}_0, \mathbf{x}_1)}{\partial x_{01}}))\ldots) \tag{25}$$

is the m-fold derivative of this function with regard to $\mathbf{x}_0$,

$$\min_{\mathbf{x}_0}{}^m f(\mathbf{x}_0, \mathbf{x}_1) = \min_{x_{0m}}(\ldots(\min_{x_{02}}(\min_{x_{01}} f(\mathbf{x}_0, \mathbf{x}_1)))\ldots) \tag{26}$$

the m-fold minimum of this function with regard to $\mathbf{x}_0$,

$$\max_{\mathbf{x}_0}{}^m f(\mathbf{x}_0, \mathbf{x}_1) = \max_{x_{0m}}(\ldots(\max_{x_{02}}(\max_{x_{01}} f(\mathbf{x}_0, \mathbf{x}_1)))\ldots) \tag{27}$$

the m-fold maximum of this function with regard to $\mathbf{x}_0$, and

$$\Delta_{\mathbf{x}_0} f(\mathbf{x}_0, \mathbf{x}_1) = \min_{\mathbf{x}_0}{}^m f(\mathbf{x}_0, \mathbf{x}_1) \oplus \max_{\mathbf{x}_0}{}^m f(\mathbf{x}_0, \mathbf{x}_1) \tag{28}$$

the Δ - operation of the Boolean function with regard to $\mathbf{x}_0$.

The vectorial derivative operations verify the change behavior of the given Boolean function depending on the simultaneous change of the values of all variables of the vector $\mathbf{x}_0$ where $\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_1)$.

Definition 3: Let $f(\mathbf{x}) = f(\mathbf{x}_0, \mathbf{x}_1)$ be a Boolean function of n variables. Then

$$\frac{\partial f(\mathbf{x}_0, \mathbf{x}_1)}{\partial \mathbf{x}_0} = f(\mathbf{x}_0, \mathbf{x}_1) \oplus f(\overline{\mathbf{x}}_0, \mathbf{x}_1) \tag{29}$$

is the vectorial derivative of this function with regard to $\mathbf{x}_0$,

$$\min_{\mathbf{x}_0} f(\mathbf{x}_0, \mathbf{x}_1) = f(\mathbf{x}_0, \mathbf{x}_1) \wedge f(\overline{\mathbf{x}}_0, \mathbf{x}_1) \tag{30}$$

the vectorial minimum of this function with regard to $\mathbf{x}_0$, and

$$\max_{\mathbf{x}_0} f(\mathbf{x}_0, \mathbf{x}_1) = f(\mathbf{x}_0, \mathbf{x}_1) \vee f(\overline{\mathbf{x}}_0, \mathbf{x}_1) \tag{31}$$

the vectorial maximum of this function with regard to $\mathbf{x}_0$.

III. Dependencies of Derivative Operations on their Direction of Change

It follows directly from Definition 1 that the result of each simple derivative operation does not depend on the variable of their direction of change:

$$\frac{\partial}{\partial x_i}\left(\frac{\partial f(\mathbf{x})}{\partial x_i}\right) = 0 \ , \tag{32}$$

$$\frac{\partial}{\partial x_i}\left(\min_{x_i} f(\mathbf{x})\right) = 0 \ , \tag{33}$$

$$\frac{\partial}{\partial x_i}\left(\max_{x_i} f(\mathbf{x})\right) = 0 \ . \tag{34}$$

Consequently, the result of each m-fold derivative operation does not depend on all variables $x_i \in \mathbf{x}_0$ which belong to their direction of change:

$$\frac{\partial}{\partial x_i}\left(\frac{\partial^m f(\mathbf{x}_0, \mathbf{x}_1)}{\partial x_{01} \partial x_{02} \ldots \partial x_{0m}}\right) = 0 \ , \tag{35}$$

$$\frac{\partial}{\partial x_i}\left(\min_{\mathbf{x}_0}{}^m f(\mathbf{x}_0, \mathbf{x}_1))\right) = 0 \ , \tag{36}$$

$$\frac{\partial}{\partial x_i}\left(\max_{\mathbf{x}_0}{}^m f(\mathbf{x}_0, \mathbf{x}_1)\right) = 0 \ . \tag{37}$$

As opposed to these derivative operations all vectorial derivative operations generally depend on all variables $x_i \in (\mathbf{x}_0, \mathbf{x}_1)$. However, it follows also directly from Definition 3 that the result of each vectorial derivative operation does not depend on the simultaneous change of the values of the same variables:

$$\frac{\partial}{\partial \mathbf{x}_0}\left(\frac{\partial f(\mathbf{x}_0, \mathbf{x}_1)}{\partial \mathbf{x}_0}\right) = 0 \ , \tag{38}$$

$$\frac{\partial}{\partial \mathbf{x}_0}\left(\min_{\mathbf{x}_0} f(\mathbf{x}_0, \mathbf{x}_1))\right) = 0 \ , \tag{39}$$

$$\frac{\partial}{\partial \mathbf{x}_0}\left(\max_{\mathbf{x}_0} f(\mathbf{x}_0, \mathbf{x}_1)\right) = 0 \ . \tag{40}$$

All n variables x_i of a Boolean space B^n are linearly independent on each other. Hence, the information about the dependency of $f(\mathbf{x})$ on the n variables can simply be stored in an *independency vector IDV* of n bits where the value 1 of bit j indicates that $f(\mathbf{x})$ does not depend on the associated variable x_j.

An independency vector cannot store the independence of the common change of several variables as used in the vectorial derivative operations. Therefore, we define the *independency matrix IDM*. The Boolean space B^n contains 2^n different Boolean vectors. Hence, there are $2^n - 1$ different directions of change starting from one of these vectors to reach each other vector. Not all of these directions of change are independent on each other. The number of independent directions of change of the Boolean space B^n is restricted to n.

Definition 4: The *independency matrix IDM* of a Boolean function $f(x_1, x_2, \ldots, x_n)$ is a Boolean matrix of n rows and n columns. The columns of the independency matrix are associated to the n variables of the Boolean space in the fixed order $(x_1, x_2, \ldots, x_n)$. The independency matrix has the shape of an echelon - all elements below the main diagonal are equal to 0, and elements above the main diagonal can be equal to 1 only if the element in the main diagonal of the same row is equal to 1. The independency matrix of a Boolean function $f(x_1, x_2, \ldots, x_n)$ must describe all independent directions of change in a unique manner.

The independency vector IDV can be mapped into the independency matrix such that the elements of the independency vector are filled into the main diagonal of the independency matrix which contains 0-values in all other positions.

Definition 5: The $\mathbf{rank}$ of an independency matrix IDM(f) describes the number of independent directions of change of the Boolean function $f(x_1, x_2, \ldots, x_n)$. The $\mathbf{rank}$(IDM(f)) is equal to the number of elements 1 in the main diagonal of the unique echelon shape of IDM(f).

Example 1: Assume the function $f(\mathbf{x})$ or a lattice of such functions depend on n variables $(x_1, x_2 \ldots, x_n)$. The associated IDM(f) is a zero matrix of n rows and n columns so that $\mathbf{rank}$(IDM$(f)) = 0$. The element IDM$(f)[i,i] = 0$ in the main diagonal of IDM must be replaced by IDM$(f)[i,i] = 1$ if the function $f(\mathbf{x})$ or all functions of a lattice of functions $f(\mathbf{x})$ do not depend on x_i. After this replacement we have $\mathbf{rank}$(IDM$(f)) = 1$, and it can easily be verified that the function or the lattice associated with IDM(f) do not depend on x_i. The condition for the independence of f on x_i is that the i-th row of IDM(f) contains the value 1 only in the position i. Similarly further values 0 can be replaced by values 1 in the main diagonal of IDM(f) for simple derivative operations with regard to other variables $x_j \neq x_i$. In case of an m-fold derivative operation several values 0 are replaced by values 1 in the main diagonal of IDM(f).

While both simple derivative operations and m-fold derivative operations change only the main diagonal of IDM(f), vectorial derivative operations change also other elements of IDM(f). Before we give an algorithm to create a unique independency matrix IDM(f), we explain the problem and the key idea based on a simple example.

Example 2: Assume that the vectorial minimum

$$g_2(\mathbf{x}) = \min_{(x_1,x_3,x_4)} f(x_1.x_2, x_3, x_4, x_5) \tag{41}$$

was calculated of the function $f(\mathbf{x})$ that depend on all 5 variables x_i. Due to (39) it holds that

$$\frac{\partial}{\partial(x_1, x_3, x_4)} (g_2(\mathbf{x})) = 0 \ . \tag{42}$$

The restriction (42) is indicated in the first row of IDM(g_2) by values 1 in the columns 1, 3, and 4. We assume furthermore that the vectorial derivative

$$g_1(\mathbf{x}) = \frac{\partial g_2(\mathbf{x})}{\partial(x_3, x_5)} \tag{43}$$

will be calculated next. Hence, $g_1(\mathbf{x})$ satisfies, due to (38), the restriction

$$\frac{\partial}{\partial(x_3, x_5)} (g_1(\mathbf{x})) = 0 \ . \tag{44}$$

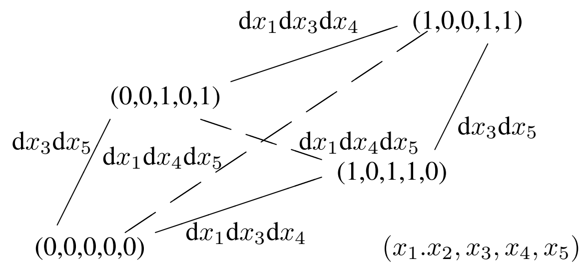

Fig. 1. Partial Boolean space of equal function values

Figure 1 shows one of the partial spaces of equal function values selected by $(x_1.x_2, x_3, x_4, x_5) = (0, 0, 0, 0, 0)$ and restricted by both (42) and (44).

The solid lines in Figure 1 indicate the evaluated directions of change of the real sequence of executed vectorial derivative operations. It can be seen that these four edges select a complete Boolean space B^2 of four nodes. The dashed lines in Figure 1 indicate an alternative direction of change of the same Boolean space. Hence, the Boolean space of Figure 1 can be defined by three different pairs of directions of change:

1) $\mathrm{d}x_1\mathrm{d}x_3\mathrm{d}x_4$ and $\mathrm{d}x_3\mathrm{d}x_5$
2) $\mathrm{d}x_1\mathrm{d}x_3\mathrm{d}x_4$ and $\mathrm{d}x_1\mathrm{d}x_4\mathrm{d}x_5$, or
3) $\mathrm{d}x_1\mathrm{d}x_4\mathrm{d}x_5$ and $\mathrm{d}x_3\mathrm{d}x_5$.

Which pair of directions of change should be selected for the unique description in the independency matrix IDM(g_1)? An easy way for this selection is the utilization of the decimal equivalent of binary row vectors of IDM(g_1) where a value 1 indicates the associated variable to change. For the studied example we get:

- $(10110)_{dez}$ =22 for $\mathrm{d}x_1\mathrm{d}x_3\mathrm{d}x_4$,
- $(10011)_{dez}$ =19 for $\mathrm{d}x_1\mathrm{d}x_4\mathrm{d}x_5$, and
- $(00101)_{dez} =$ 5 for $\mathrm{d}x_3\mathrm{d}x_5$.

It can be seen that the EXOR of any pair of these vectors is equal to the remaining vector. Hence, each of these three pairs describes two independent directions of changes. These directions of change can be separated in a first step regarding the most significant bit that is equal to 1 into the subsets of vectors $\{(10110)_{dez} = 22$ or $(10011)_{dez} = 19\}$ and $\{(00101)_{dez} = 5\}$. As a unique selection we use for such a subset of directions of changes the vector with the smallest decimal equivalent which is $(10011)_{dez} = 19$ for the first subset. Hence, the uniquely chosen direction vectors are in this example (10011) and (00101). These vectors are written into the independency matrix IDM(g_1) such that the most significant 1-bit belongs to the main diagonal.

$$\text{IDM}(g_1(x_1.x_2, x_3, x_4, x_5)) = \begin{vmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{vmatrix} \tag{45}$$

Example 3: Using the unique IDM(g_1), it is easy to check that (10110) is not a new independent direction of change, because the most significant bit of (10110) occurs in the main diagonal of IDM(g_1). The EXOR of this vector with the first line of IDM(g_1) describes the remaining direction of change:

$$(10110) \oplus (10011) = (00101) \ . \tag{46}$$

In the resulting vector (46) the most significant bit is in the third position. Hence, the EXOR of the resulting vector (46) with the third line of IDM(g_1) (which is (00101)) is built:

$$(00101) \oplus (00101) = (00000) \ . \tag{47}$$

The resulting vector (00000) of (47) indicates that the function $g_1(x_1.x_2, x_3, x_4, x_5)$ does not depend on the simultaneous change of the values of x_1, x_3, x_4:

$$\frac{\partial}{\partial(x_1, x_3, x_4)}(g_1(x_1.x_2, x_3, x_4, x_5)) = 0 \ . \tag{48}$$

A row of IDM(f) indicates by values 1 a subset of variables used in a vectorial derivative operation. The relation between a subset of variables $\mathbf{x}_0$ and the associated binary vector $\mathbf{s}_0$ can be defined as follows:

Definition 6: Let $f(\mathbf{x}) = f(x_1, x_2, \ldots, x_n)$ be a Boolean function and $\mathbf{x}_0 \in \mathbf{x}$ a subset of variables; then

$$\mathbf{s}_0 = BV(\mathbf{x}_0) \tag{49}$$

is a *binary vector* of n elements where $\mathbf{s}_0[i] = 1$ indicates that $x_i \in \mathbf{x}_0$.

Two more important questions arise:

1) How can it generally be verified whether the function $f(\mathbf{x})$ of a given unique independency matrix depends on the variables $\mathbf{x}_0$ with $\mathbf{s}_0 = BV(\mathbf{x}_0)$?
2) How can a minimal vector $\mathbf{s}_{min}$ be found which indicates by values 1 the minimal direction of change of a given subset of variables of $\mathbf{x}_0$ and a unique independency matrix IDM(f)?

Algorithm 1 calculates the binary vector $\mathbf{s}_{min}$. This Algorithm iterates over the columns j and uses:

- $\mathbf{s}_{min}[j]$ the binary value of $\mathbf{s}_{min}$ on the position j,
- IDM$(f)[j, j]$ the element of the row j and the column j in IDM(f),
- IDM$(f)[j]$ the row j of IDM(f).

Algorithm 1 $\mathbf{s}_{min} =$ MIDC(IDM$(f), \mathbf{x}_0$): Minimal Independent Direction of Change

Require: $\mathbf{x}_0 \in \mathbf{x}$: evaluated subset of variables, IDM(f): unique independency matrix of n rows and n columns
Ensure: $\mathbf{s}_{min}$: minimal direction of change

1: $j \leftarrow 1$
2: $\mathbf{s}_{min} \leftarrow BV(\mathbf{x}_0)$
3: **while** $j < (n+1)$ **do**
4:     **if** $(\mathbf{s}_{min}[j] = 1) \wedge (IDM(f)[j,j] = 1)$ **then**
5:         $\mathbf{s}_{min} \leftarrow \mathbf{s}_{min} \oplus$ IDM$(f)[j]$
6:     **end if**
7:     $j \leftarrow j + 1$
8: **end while**

Theorem 1: The function $f(\mathbf{x})$ with the independency matrix IDM(f) does not depend on the variables of $\mathbf{x}_0$ if

$$\mathbf{s}_{min} = \text{MIDC(IDM}(f), \mathbf{x}_0) = 0 \ . \tag{50}$$

Proof: The EXOR-operations of $BV(\mathbf{x}_0)$ with such rows of IDM(f) selected by line 4 of Algorithm 1 creates an empty vector $\mathbf{s}_{min}$. Hence, the direction of change defined by $\mathbf{x}_0$ matches a sequence of directions of changes defined by IDM(f). ■

The task of the second question is solved by Algorithm 2. The independency matrix IDM(g) is equal to IDM(f) if f is independent on the variables $\mathbf{x}_0$. Otherwise, the independency matrix IDM(g) is constructed in a unique manner because it contains in the column of the most significant bit of $\mathbf{s}_{min}$ only 0-elements except the main diagonal where $\mathbf{s}_{min}$ is included into the independency matrix IDM(g).

Algorithm 2 IDM$(g) =$ UM(IDM$(f), \mathbf{x}_0$): Unique Merge

Require: $\mathbf{x}_0 \in \mathbf{x}$: subset of variables to merge with IDM(f), IDM(f): unique independency matrix of n rows and n columns of the Boolean function f
Ensure: IDM(g): unique independency matrix of n rows and n columns of the function g which is the result of a vectorial derivative operation with regard to the variables $\mathbf{x}_0$

IDM$(g) \leftarrow$ IDM(f)
$\mathbf{s}_{min} =$ MIDC(IDM$(f), \mathbf{x}_0$)
if $\mathbf{s}_{min} > 0$ **then**
    $j \leftarrow$ IndexOfMostSignificantBit($\mathbf{s}_{min}$)
    $i \leftarrow 1$
    while $i < n$ **do**
        if IDM$(g)[i,j] = 1$ **then**
            IDM$(g)[i] \leftarrow$ IDM$(g)[i] \oplus \mathbf{s}_{min}$
        end if
        $i \leftarrow i + 1$
    end while
    IDM$(g)[j] \leftarrow \mathbf{s}_{min}$
end if

The application of the independency matrix IDM(f) and the associated $\mathbf{rank}$ will be discussed in the next section more in detail.

IV. Lattices of Boolean Functions in B^n

Most generally all Boolean functions $f(\mathbf{x})$ of a lattice can depend on all n variables $(x_1, x_2, \ldots, x_n)$. Using the restrictive equations (11) and (12) instead of the inequality (10) a single restrictive Boolean equation (51) can specify all properties of such a lattice:

$$f_q(\mathbf{x}) \wedge \overline{f(\mathbf{x})} \vee f(\mathbf{x}) \wedge f_r(\mathbf{x}) = 0 \ . \tag{51}$$

More general lattices exist besides this special lattice of Boolean functions. Some properties of all functions of the lattice can be that all these functions do not depend on one variable, on several variables or on the simultaneous changes of the values of several variables.

There are 2^{2^n} different Boolean functions in the Boolean space B^n. Not all of them really depend on all n variables. As example we transform the simple derivative (22) using the Shannon expansion and equivalent transformations from the Boolean algebra:

$$\begin{aligned}\frac{\partial f(x_i,\mathbf{x}_1)}{\partial x_i} =& f(x_i,\mathbf{x}_1) \oplus f(\overline{x}_i,\mathbf{x}_1) \\ =& x_i \wedge f(x_i=1,\mathbf{x}_1) \oplus \overline{x}_i \wedge f(x_i=0,\mathbf{x}_1) \oplus \\ & x_i \wedge f(\overline{x}_i=1,\mathbf{x}_1) \oplus \overline{x}_i \wedge f(\overline{x}_i=0,\mathbf{x}_1) \\ =& x_i \wedge f(x_i=1,\mathbf{x}_1) \oplus \overline{x}_i \wedge f(x_i=0,\mathbf{x}_1) \oplus \\ & x_i \wedge f(x_i=0,\mathbf{x}_1) \oplus \overline{x}_i \wedge f(x_i=1,\mathbf{x}_1) \\ =& x_i \wedge f(x_i=1,\mathbf{x}_1) \oplus \overline{x}_i \wedge f(x_i=1,\mathbf{x}_1) \oplus \\ & x_i \wedge f(x_i=0,\mathbf{x}_1) \oplus \overline{x}_i \wedge f(x_i=0,\mathbf{x}_1) \\ =& (x_i \oplus \overline{x}_i) \wedge f(x_i=1,\mathbf{x}_1) \oplus \\ & (x_i \oplus \overline{x}_i) \wedge f(x_i=0,\mathbf{x}_1) \\ =& f(x_i=1,\mathbf{x}_1) \oplus f(x_i=0,\mathbf{x}_1)\ . \end{aligned} \tag{52}$$

The transformation (52) shows that the simple derivative of the Boolean function $f(x_i,\mathbf{x}_1)$ with regard to x_i can be calculated as EXOR of the positive cofactor and the negative cofactor of the given function. The value of the variables x_i is fixed to a constant in both cofactors. Hence, the result of the simple derivatives of $f(x_i,\mathbf{x}_1)$ with regard to x_i does not depend on the variable x_i.

The knowledge that a function does not depend on one variable or several variables can help to simplify the task to solve. Hence this information should be stored together with the function itself. A simple possibility to remember this information is the Boolean differential equation

$$\frac{\partial f(\mathbf{x})}{\partial x_i} = 0 \tag{53}$$

which describes that the function $f(\mathbf{x})$ does not depend on the variable x_i.

As explored for a single function all functions of a lattice of Boolean functions can be independent on the variable x_i. Using the restrictive equations (11) and (12) instead of the inequality (10) a single restrictive Boolean equation (54) can specify all properties of the lattice:

$$f_q(\mathbf{x}) \wedge \overline{f(\mathbf{x})} \vee f(\mathbf{x}) \wedge f_r(\mathbf{x}) \vee \frac{\partial f(\mathbf{x})}{\partial x_i} = 0\ . \tag{54}$$

The first two terms in (54) are needed for each lattice of Boolean functions of B^n. The mark functions $f_q(\mathbf{x})$ and $f_r(\mathbf{x})$ in (54) satisfy the restriction (7). The last term in (54) does not appear or can be replaced by the constant function 0 if the functions of the lattice depend on all n variables of B^n.

A stronger restriction can be that all functions of a lattice do not depend on all variables $x_i \in \mathbf{x_0}$. These properties can be expressed also using a single slightly extended restrictive Boolean equation: (55):

$$f_q(\mathbf{x}) \wedge \overline{f(\mathbf{x})} \vee f(\mathbf{x}) \wedge f_r(\mathbf{x}) \vee \bigvee_{x_i \in \mathbf{x_0}} \frac{\partial f(\mathbf{x})}{\partial x_i} = 0\ . \tag{55}$$

The variables x_i of the last term of (55) can be stored in an independency matrix IDM(f) which includes the corresponding values 1 only in the main diagonal. The number of variables in the set of variables $\mathbf{x_0}$ is equal to $\mathbf{rank}(\text{IDM}(f))$.

Most generally, a lattice can contain only functions which do not change their values in the case of the simultaneous change of the values of several variables or even a set of directions of change. These directions of change can be expressed by a disjunction of appropriate vectorial derivatives which are uniquely indicated in the independency matrix. For a short notation we define an *independency function* $f^{id}(\mathbf{x})$.

Definition 7: The *independency function* $f^{id}(x_1,\ldots,x_n)$ of a Boolean function corresponds to the independency matrix IDM(f) such that

$$f^{id}(x_1,\ldots,x_n) = f^{id}(\mathbf{x}) = \bigvee_{i=1}^{n} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}_{0i}} \tag{56}$$

where

$$\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}_{0i}} = 0 \tag{57}$$

if all elements of the row i in IDM(f) are equal to 0, and

$$x_j \in \mathbf{x}_{0i} \text{ if IDM}(f)[i,j] = 1. \tag{58}$$

Using Definition 7 we get the most general description of a lattice of Boolean functions $f(x_1,x_2,\ldots,x_n)$ which do not depend on all directions of change indicated in the associated unique independency matrix IDM(f) by the restrictive Boolean equation (59)

$$f_q(\mathbf{x}) \wedge \overline{f(\mathbf{x})} \vee f(\mathbf{x}) \wedge f_r(\mathbf{x}) \vee f^{id}(\mathbf{x}) = 0\ . \tag{59}$$

As strongest restriction of a lattice regarding the dependencies of the variables no function of the lattice depends on any of the n variables of B^n. Such a lattice can contain only the constant functions $f(\mathbf{x}) = 1$ and $f(\mathbf{x}) = 0$. Both (55) and (59) can be restricted to this case such that the vector $\mathbf{x_0}$ in the last term of (55) is replaced by the vector of all variables $\mathbf{x}$ or the IDM(f) corresponding to $f^{id}(\mathbf{x})$ of (59) is replaced by an identity matrix with $\mathbf{rank}(\text{IDM}(f)) = n$.

V. Derivative Operations for Lattices of Boolean Functions

The independency matrix IDM(f) of a lattice of Boolean functions describes the independency of all functions on certain directions of change and influences the result of all derivatives operations due to (32), …, (40).

Theorem 2: Let $f(\mathbf{x}) = f(x_1,x_2,\ldots,x_n)$ be a Boolean function of n variables that belongs to the lattice defined by the equation (59) where $f_q(\mathbf{x})$ and $f_r(\mathbf{x})$ satisfy (7), and $f(\mathbf{x})$ depends on the simultaneous change of the values of all variables of $\mathbf{x}_0$:

$$\text{MIDC}(\text{IDM}(f),\mathbf{x}_0) > 0\ . \tag{60}$$

Then all vectorial derivatives

$$g_1(\mathbf{x}) = \frac{\partial f(\mathbf{x}_0,\mathbf{x}_1)}{\partial \mathbf{x}_0} \tag{61}$$

belong to a Boolean lattice defined by

$$f_q^{\partial \mathbf{x}_0}(\mathbf{x}_0,\mathbf{x}_1) \wedge \overline{g_1(\mathbf{x})}$$
$$\vee g_1(\mathbf{x}) \wedge f_r^{\partial \mathbf{x}_0}(\mathbf{x}_0,\mathbf{x}_1) \vee g_1^{id}(\mathbf{x}) = 0 \qquad (62)$$

with the mark functions of the vectorial derivative of the lattice with regard to $\mathbf{x}_0$

$$f_q^{\partial \mathbf{x}_0}(\mathbf{x}_0,\mathbf{x}_1) = \max_{\mathbf{x}_0} f_q(\mathbf{x}_0,\mathbf{x}_1) \wedge \max_{\mathbf{x}_0} f_r(\mathbf{x}_0,\mathbf{x}_1)\ , \qquad (63)$$
$$f_r^{\partial \mathbf{x}_0}(\mathbf{x}_0,\mathbf{x}_1) = \min_{\mathbf{x}_0} f_q(\mathbf{x}_0,\mathbf{x}_1) \vee \min_{\mathbf{x}_0} f_r(\mathbf{x}_0,\mathbf{x}_1)\ , \qquad (64)$$

and the independency function $g_1^{id}(\mathbf{x})$ associated to

$$\text{IDM}(g_1) = \text{UM}(\text{IDM}(f),\mathbf{x}_0)\ ; \qquad (65)$$

all vectorial minimums

$$g_2(\mathbf{x}) = \min_{\mathbf{x}_0} f(\mathbf{x}) \qquad (66)$$

belong to a Boolean lattice defined by

$$f_q^{\min_{\mathbf{x}_0}}(\mathbf{x}_0,\mathbf{x}_1) \wedge \overline{g_2(\mathbf{x})}$$
$$\vee g_2(\mathbf{x}) \wedge f_r^{\min_{\mathbf{x}_0}}(\mathbf{x}_0,\mathbf{x}_1) \vee g_2^{id}(\mathbf{x}) = 0 \qquad (67)$$

with the mark functions of the vectorial minimum of the lattice with regard to $\mathbf{x}_0$

$$f_q^{\min_{\mathbf{x}_0}}(\mathbf{x}_0,\mathbf{x}_1) = \min_{\mathbf{x}_0} f_q(\mathbf{x}_0,\mathbf{x}_1)\ , \qquad (68)$$
$$f_r^{\min_{\mathbf{x}_0}}(\mathbf{x}_0,\mathbf{x}_1) = \max_{\mathbf{x}_0} f_r(\mathbf{x}_0,\mathbf{x}_1)\ , \qquad (69)$$

and the independency function $g_2^{id}(\mathbf{x})$ associated to

$$\text{IDM}(g_2) = \text{UM}(\text{IDM}(f),\mathbf{x}_0)\ ; \qquad (70)$$

and all vectorial maximums

$$g_3(\mathbf{x}) = \max_{\mathbf{x}_0} f(\mathbf{x}) \qquad (71)$$

belong to a Boolean lattice defined by

$$f_q^{\max_{\mathbf{x}_0}}(\mathbf{x}_0,\mathbf{x}_1) \wedge \overline{g_3(\mathbf{x})}$$
$$\vee g_3(\mathbf{x}) \wedge f_r^{\max_{\mathbf{x}_0}}(\mathbf{x}_0,\mathbf{x}_1) \vee g_3^{id}(\mathbf{x}) = 0 \qquad (72)$$

with the mark functions of the simple maximum of the lattice with regard to $\mathbf{x}_0$

$$f_q^{\max_{\mathbf{x}_0}}(\mathbf{x}_0,\mathbf{x}_1) = \max_{\mathbf{x}_0} f_q(\mathbf{x}_0,\mathbf{x}_1)\ , \qquad (73)$$
$$f_r^{\max_{\mathbf{x}_0}}(\mathbf{x}_0,\mathbf{x}_1) = \min_{\mathbf{x}_0} f_r(\mathbf{x}_0,\mathbf{x}_1)\ , \qquad (74)$$

and the independency function $g_3^{id}(\mathbf{x})$ associated to

$$\text{IDM}(g_3) = \text{UM}(\text{IDM}(f),\mathbf{x}_0)\ . \qquad (75)$$

The three independency functions are equal to each other:

$$g_1^{id}(\mathbf{x}) = g_2^{id}(\mathbf{x}) = g_3^{id}(\mathbf{x}) = g^{id}(\mathbf{x})\ , \qquad (76)$$

with

$$\text{IDM}(g) = \text{UM}(\text{IDM}(f),\mathbf{x}_0)\ . \qquad (77)$$

Proof: The Boolean equation (59) can be split into a system of three equations:

$$f_q(\mathbf{x}) \wedge \overline{f(\mathbf{x})} = 0\ , \qquad (78)$$
$$f(\mathbf{x}) \wedge f_r(\mathbf{x}) = 0\ , \qquad (79)$$
$$f^{id}(\mathbf{x}) = 0\ . \qquad (80)$$

The premise (59) of Theorem 2 is valid for all assignments of Boolean values to the variables $\mathbf{x}$ - therefore (78) and (79) can be transformed into

$$f_q(\mathbf{x}_0,\mathbf{x}_1) \wedge \overline{f(\mathbf{x}_0,\mathbf{x}_1)} = f_q(\mathbf{x}_0) \wedge \overline{f(\mathbf{x}_0)} = 0\ , \qquad (81)$$
$$f_q(\overline{\mathbf{x}}_0,\mathbf{x}_1) \wedge \overline{f(\overline{\mathbf{x}}_0,\mathbf{x}_1)} = f_q(\overline{\mathbf{x}}_0) \wedge \overline{f(\overline{\mathbf{x}}_0)} = 0\ , \qquad (82)$$
$$f(\mathbf{x}_0,\mathbf{x}_1) \wedge f_r(\mathbf{x}_0,\mathbf{x}_1) = f(\mathbf{x}_0) \wedge f_r(\mathbf{x}_0) = 0\ , \qquad (83)$$
$$f(\overline{\mathbf{x}}_0,\mathbf{x}_1) \wedge f_r(\overline{\mathbf{x}}_0,\mathbf{x}_1) = f(\overline{\mathbf{x}}_0) \wedge f_r(\overline{\mathbf{x}}_0) = 0\ . \qquad (84)$$

The conclusion (62) for the lattice of vectorial derivatives can be transformed into a system of three equations which can be verified separately:

$$f_q^{\partial \mathbf{x}_0}(\mathbf{x}_0,\mathbf{x}_1) \wedge \overline{g_1(\mathbf{x})} = 0\ , \qquad (85)$$
$$g_1(\mathbf{x}) \wedge f_r^{\partial \mathbf{x}_0}(\mathbf{x}_1) = 0\ , \qquad (86)$$
$$g_1^{id}(\mathbf{x}) = 0\ . \qquad (87)$$

Using the definitions

1) of $g_1(\mathbf{x})$ (61),
2) of the vectorial derivative (29),
3) of the mark functions of the vectorial derivatives (63) and (64),
4) of the vectorial minimum (30),
5) of the vectorial maximum (31), and
6) the short notation introduced in (81), (82), (83), and (84),

we get for (85)

$$(f_q(\mathbf{x}_0) \vee f_q(\overline{\mathbf{x}}_0)) \wedge (f_r(\mathbf{x}_0) \vee f_r(\overline{\mathbf{x}}_0)) \wedge$$
$$\overline{(f(\mathbf{x}_0) \oplus f(\overline{\mathbf{x}}_0)} = 0\ , \qquad (88)$$

which is equal to

$$(f_q(\mathbf{x}_0) \vee f_q(\overline{\mathbf{x}}_0)) \wedge (f_r(\mathbf{x}_0) \vee f_r(\overline{\mathbf{x}}_0)) \wedge$$
$$((f(\mathbf{x}_0) \wedge f(\overline{\mathbf{x}}_0) \vee \overline{f(\mathbf{x}_0)} \wedge \overline{f(\overline{\mathbf{x}}_0)}) = 0\ , \qquad (89)$$

and can be transformed into a disjunctive form that contains eight conjunctions:

$$f_q(\mathbf{x}_0) \wedge f_r(\mathbf{x}_0) \wedge f(\mathbf{x}_0) \wedge f(\overline{\mathbf{x}}_0)$$
$$\vee f_q(\mathbf{x}_0) \wedge f_r(\mathbf{x}_0) \wedge \overline{f(\mathbf{x}_0)} \wedge \overline{f(\overline{\mathbf{x}}_0)}$$
$$\vee f_q(\mathbf{x}_0) \wedge f_r(\overline{\mathbf{x}}_0) \wedge f(\mathbf{x}_0) \wedge f(\overline{\mathbf{x}}_0)$$
$$\vee f_q(\mathbf{x}_0) \wedge f_r(\overline{\mathbf{x}}_0) \wedge \overline{f(\mathbf{x}_0)} \wedge \overline{f(\overline{\mathbf{x}}_0)}$$
$$\vee f_q(\overline{\mathbf{x}}_0) \wedge f_r(\mathbf{x}_0) \wedge f(\mathbf{x}_0) \wedge f(\overline{\mathbf{x}}_0)$$
$$\vee f_q(\overline{\mathbf{x}}_0) \wedge f_r(\mathbf{x}_0) \wedge \overline{f(\mathbf{x}_0)} \wedge \overline{f(\overline{\mathbf{x}}_0)}$$
$$\vee f_q(\overline{\mathbf{x}}_0) \wedge f_r(\overline{\mathbf{x}}_0) \wedge f(\mathbf{x}_0) \wedge f(\overline{\mathbf{x}}_0)$$
$$\vee f_q(\overline{\mathbf{x}}_0) \wedge f_r(\overline{\mathbf{x}}_0) \wedge \overline{f(\mathbf{x}_0)} \wedge \overline{f(\overline{\mathbf{x}}_0)} = 0\ . \qquad (90)$$

All conjunctions in (90) are equal to 0 due to

- (81) for the conjunctions 2 and 4,

- (82) for the conjunctions 6 and 8,
- (83) for the conjunctions 1 and 5, and
- (84) for the conjunctions 3 and 7

which proves the first conclusion (85). Similarly we get for the second conclusion (86)

$$\begin{aligned}(f(\mathbf{x}_0) \oplus f(\overline{\mathbf{x}}_0))\wedge \\ [(f_q(\mathbf{x}_0) \wedge f_q(\overline{\mathbf{x}}_0)) \vee (f_r(\mathbf{x}_0) \wedge f_r(\overline{\mathbf{x}}_0))] = 0 \;,\end{aligned} \tag{91}$$

which is equal to

$$\begin{aligned}(f(\mathbf{x}_0) \wedge \overline{f(\overline{\mathbf{x}}_0)} \vee \overline{f(\mathbf{x}_0)} \wedge f(\overline{\mathbf{x}}_0))\wedge \\ [(f_q(\mathbf{x}_0) \wedge f_q(\overline{\mathbf{x}}_0)) \vee (f_r(\mathbf{x}_0) \wedge f_r(\overline{\mathbf{x}}_0))] = 0 \;,\end{aligned} \tag{92}$$

and can be transformed into a disjunctive form that contains four conjunctions:

$$\begin{aligned}&f(\mathbf{x}_0) \wedge \overline{f(\overline{\mathbf{x}}_0)} \wedge f_q(\mathbf{x}_0) \wedge f_q(\overline{\mathbf{x}}_0) \\ \vee &f(\mathbf{x}_0) \wedge \overline{f(\overline{\mathbf{x}}_0)} \wedge f_r(\mathbf{x}_0) \wedge f_r(\overline{\mathbf{x}}_0) \\ \vee &\overline{f(\mathbf{x}_0)} \wedge f(\overline{\mathbf{x}}_0) \wedge f_q(\mathbf{x}_0) \wedge f_q(\overline{\mathbf{x}}_0) \\ \vee &\overline{f(\mathbf{x}_0)} \wedge f(\overline{\mathbf{x}}_0) \wedge f_r(\mathbf{x}_0) \wedge f_r(\overline{\mathbf{x}}_0) = 0 \;.\end{aligned} \tag{93}$$

All conjunctions in (93) are equal to 0 due to

- (81) for the conjunction 3,
- (82) for the conjunction 1,
- (83) for the conjunction 2, and
- (84) for the conjunction 4,

which proves the second conclusion (86). The third conclusion (87) follows from (38) and the construction of the associated independency matrix (65).

Hence, the proof that the vectorial derivatives of all functions of (59) belong to the new lattice (62) with the mark functions (63) and (64) and the independency function $g_1^{id}(\mathbf{x})$ associated to (65) is complete.

It follows the proof that all functions $g_2(\mathbf{x})$ (66) belong to the Boolean lattice (67) with the mark functions (68) and (69) and the independency function $g_2^{id}(\mathbf{x})$ associated to (70).

The premise for the lattice of vectorial maximums is the same as for vectorial derivatives (81), (82), (83), and (84). The conclusion (67) for the lattice of vectorial minimums can be also transformed into a system of three equations which can be verified separately:

$$f_q^{\min_{\mathbf{x}_0}}(\mathbf{x}_0, \mathbf{x}_1) \wedge \overline{g_2(\mathbf{x})} = 0 \;, \tag{94}$$

$$g_2(\mathbf{x}) \wedge f_r^{\min_{\mathbf{x}_0}}(\mathbf{x}_0, \mathbf{x}_1) = 0 \;, \tag{95}$$

$$g_2^{id}(\mathbf{x}) = 0 \;. \tag{96}$$

Using the definitions

1) of $g_2(\mathbf{x})$ (66),
2) of the vectorial minimum (30),
3) of the mark functions of the vectorial minimums (68) and (69),
4) of the vectorial maximum (31), and
5) the short notation introduced in (81), (82), (83), and (84),

we get for (94)

$$(f_q(\mathbf{x}_0) \wedge f_q(\overline{\mathbf{x}}_0)) \wedge \overline{(f(\mathbf{x}_0) \wedge f(\overline{\mathbf{x}}_0))} = 0 \;, \tag{97}$$

which is equal to

$$(f_q(\mathbf{x}_0) \wedge f_q(\overline{\mathbf{x}}_0)) \wedge (\overline{f(\mathbf{x}_0)} \vee \overline{f(\overline{\mathbf{x}}_0)}) = 0 \;, \tag{98}$$

and can be transformed into a disjunctive form that contains two conjunctions:

$$\begin{aligned}&f_q(\mathbf{x}_0) \wedge f_q(\overline{\mathbf{x}}_0) \wedge \overline{f(\mathbf{x}_0)} \\ \vee &f_q(\mathbf{x}_0) \wedge f_q(\overline{\mathbf{x}}_0) \wedge \overline{f(\overline{\mathbf{x}}_0)} = 0 \;.\end{aligned} \tag{99}$$

All conjunctions in (99) are equal to 0 due to

- (81) for the conjunction 1, and
- (82) for the conjunction 2,

which proves the first conclusion (94) for the vectoral minimum. Similarly we get for the second conclusion (95)

$$(f(\mathbf{x}_0) \wedge f(\overline{\mathbf{x}}_0)) \wedge (f_r(\mathbf{x}_0) \vee f_r(\overline{\mathbf{x}}_0)) = 0 \;, \tag{100}$$

which can be transformed into a disjunctive form that contains four conjunctions:

$$\begin{aligned}&f(\mathbf{x}_0) \wedge f(\overline{\mathbf{x}}_0) \wedge f_r(\mathbf{x}_0) \\ \vee &f(\mathbf{x}_0) \wedge f(\overline{\mathbf{x}}_0) \wedge f_r(\overline{\mathbf{x}}_0) = 0 \;.\end{aligned} \tag{101}$$

All conjunctions in (101) are equal to 0 due to

- (83) for the conjunction 1, and
- (84) for the conjunction 2,

which proves the second conclusion (95). The third conclusion (96) follows from (39) and the construction of the associated independency matrix (70).

Hence, the proof for the vectorial minimums of all functions of (59) to the new lattice (67) with the mark functions (68) and (69) and the independency function $g_2^{id}(\mathbf{x})$ associated to (70) is complete.

Finally we prove that all functions $g_3(\mathbf{x})$ (71) belong to the Boolean lattice (72) with the mark functions (73) and (74) and the independency function $g_3^{id}(\mathbf{x})$ associated to (75).

The premise for the lattice of vectorial maximums is the same as for vectorial derivatives (81), (82), (83), and (84). The conclusion (72) for the lattice of vectorial maximums can be also transformed into a system of three equations which can be verified separately.

$$f_q^{\max_{\mathbf{x}_0}}(\mathbf{x}_0, \mathbf{x}_1) \wedge \overline{g_3(\mathbf{x})} = 0 \;, \tag{102}$$

$$g_3(\mathbf{x}) \wedge f_r^{\max_{\mathbf{x}_0}}(\mathbf{x}_0, \mathbf{x}_1) = 0 \;, \tag{103}$$

$$g_3^{id}(\mathbf{x}) = 0 \;. \tag{104}$$

Using the definitions

1) of $g_3(\mathbf{x})$ (71),
2) of the vectorial maximum (31),
3) of the mark functions of the vectorial maximuns (73) and (74),
4) of the vectorial minimum (30), and
5) the short notation introduced in (81), (82), (83), and (84),

we get for (102)

$$(f_q(\mathbf{x}_0) \vee f_q(\overline{\mathbf{x}}_0)) \wedge \overline{(f(\mathbf{x}_0) \vee f(\overline{\mathbf{x}}_0))} = 0 \;, \tag{105}$$

which is equal to

$$(f_q(\mathbf{x}_0) \vee f_q(\overline{\mathbf{x}}_0)) \wedge (\overline{f(\mathbf{x}_0)} \wedge \overline{f(\overline{\mathbf{x}}_0)}) = 0 \ , \tag{106}$$

and can be transformed into a disjunctive form that contains two conjunctions:

$$\begin{aligned} & f_q(\mathbf{x}_0) \wedge \overline{f(\mathbf{x}_0)} \wedge \overline{f(\overline{\mathbf{x}}_0)} \\ \vee & f_q(\overline{\mathbf{x}}_0) \wedge \overline{f(\mathbf{x}_0)} \wedge \overline{f(\overline{\mathbf{x}}_0)} = 0 \ . \end{aligned} \tag{107}$$

All conjunctions in (107) are equal to 0 due to

- (81) for the conjunction 1, and
- (82) for the conjunction 2,

which proves the first conclusion (102) for the vectorial maximum. Similarly we get for the second conclusion (103)

$$(f(\mathbf{x}_0) \vee f(\overline{\mathbf{x}}_0)) \wedge (f_r(\mathbf{x}_0) \wedge f_r(\overline{\mathbf{x}}_0)) = 0 \ , \tag{108}$$

which can be transformed into a disjunctive form that contains two conjunctions:

$$\begin{aligned} & f(\mathbf{x}_0) \wedge f_r(\mathbf{x}_0) \wedge f_r(\overline{\mathbf{x}}_0) \\ \vee & f(\overline{\mathbf{x}}_0) \wedge f_r(\mathbf{x}_0) \wedge f_r(\overline{\mathbf{x}}_0) = 0 \ . \end{aligned} \tag{109}$$

All conjunctions in (109) are equal to 0 due to

- (83) for the conjunction 1, and
- (84) for the conjunction 2,

which proves the second conclusion (103). The third conclusion (104) follows from (40) and the construction of the associated independency matrix (75).

Hence, the proof for the vectorial maximums of all functions of (59) to the new lattice (72) with the mark functions (73) and (74) and the independency function $g_3^{id}(\mathbf{x})$ associated to (75) is complete.

Equation (76) holds due to the same direction of change $\mathbf{x}_0$ for all three vectorial derivative operations and (38), (39), (40) so that we have the complete Theorem 2. ■

The direction of change can be restricted for all Boolean functions of a given lattice to a single variable x_i.

Theorem 3: Let $f(\mathbf{x}) = f(x_1, x_2, \ldots, x_n)$ be a Boolean function of n variables that belongs to the lattice defined by (59) where $f_q(\mathbf{x})$ and $f_r(\mathbf{x})$ satisfy (7), and $f(\mathbf{x})$ depends on x_i:

$$\text{MIDC}(\text{IDM}(f), x_i) > 0 \ . \tag{110}$$

Then all simple derivatives

$$g_1(\mathbf{x}) = \frac{\partial f(\mathbf{x})}{\partial x_i} \tag{111}$$

belong to a Boolean lattice defined by

$$f_q^{\partial x_i}(\mathbf{x}_1) \wedge \overline{g_1(\mathbf{x})} \vee g_1(\mathbf{x}) \wedge f_r^{\partial x_i}(\mathbf{x}_1) \vee g_1^{id}(\mathbf{x}) = 0 \tag{112}$$

with the mark functions of the simple derivative of the lattice with regard to x_i

$$f_q^{\partial x_i}(\mathbf{x}_1) = \max_{x_i} f_q(x_i, \mathbf{x}_1) \wedge \max_{x_i} f_r(x_i, \mathbf{x}_1) \ , \tag{113}$$

$$f_r^{\partial x_i}(\mathbf{x}_1) = \min_{x_i} f_q(x_i, \mathbf{x}_1) \vee \min_{x_i} f_r(x_i, \mathbf{x}_1) \ , \tag{114}$$

and the independency function $g_1^{id}(\mathbf{x})$ associated to

$$\text{IDM}(g_1) = \text{UM}(\text{IDM}(f), x_i) \ ; \tag{115}$$

all simple minimums

$$g_2(\mathbf{x}) = \min_{x_i} f(\mathbf{x}) \tag{116}$$

belong to a Boolean lattice defined by

$$f_q^{\min_{x_i}}(\mathbf{x}_1) \wedge \overline{g_2(\mathbf{x})} \vee g_2(\mathbf{x}) \wedge f_r^{\min_{x_i}}(\mathbf{x}_1) \vee g_2^{id}(\mathbf{x}) = 0 \tag{117}$$

with the mark functions of the simple minimum of the lattice with regard to x_i

$$f_q^{\min_{x_i}}(\mathbf{x}_1) = \min_{x_i} f_q(x_i, \mathbf{x}_1) \ , \tag{118}$$

$$f_r^{\min_{x_i}}(\mathbf{x}_1) = \max_{x_i} f_r(x_i, \mathbf{x}_1) \ , \tag{119}$$

and the independency function $g_2^{id}(\mathbf{x})$ associated to

$$\text{IDM}(g_2) = \text{UM}(\text{IDM}(f), x_i) \ ; \tag{120}$$

and all simple maximums

$$g_3(\mathbf{x}) = \max_{x_i} f(\mathbf{x}) \tag{121}$$

belong to a Boolean lattice defined by

$$f_q^{\max_{x_i}}(\mathbf{x}_1) \wedge \overline{g_3(\mathbf{x})} \vee g_3(\mathbf{x}) \wedge f_r^{\max_{x_i}}(\mathbf{x}_1) \vee g_3^{id}(\mathbf{x}) = 0 \tag{122}$$

with the mark functions of the simple maximum of the lattice with regard to x_i

$$f_q^{\max_{x_i}}(\mathbf{x}_1) = \max_{x_i} f_q(x_i, \mathbf{x}_1) \ , \tag{123}$$

$$f_r^{\max_{x_i}}(\mathbf{x}_1) = \min_{x_i} f_r(x_i, \mathbf{x}_1) \ , \tag{124}$$

and the independency function $g_3^{id}(\mathbf{x})$ associated to

$$\text{IDM}(g_3) = \text{UM}(\text{IDM}(f), x_i) \ . \tag{125}$$

Proof: The proof of Theorem 3 is equivalent to the proof of Theorem 2 where the set of variables $\mathbf{x}_0$ is replaced by the single variable x_i. ■

Repeated derivative operations of the same type regarding different variables are summarized to m-fold derivative operations. Consequently, each m-fold derivative operation of a lattice of Boolean functions creates again a lattice of Boolean functions.

Theorem 4: Let $f(\mathbf{x}) = f(x_1, x_2, \ldots, x_n)$ be a Boolean function of n variables that belongs to the lattice defined by (59) where $f_q(\mathbf{x})$ and $f_r(\mathbf{x})$ satisfy (7), and $f(\mathbf{x})$ is not independent on all variable $x_{0i} \in \mathbf{x}_0$.
Then all m-fold derivatives

$$g_1(\mathbf{x}) = \frac{\partial^m f(\mathbf{x}_0, \mathbf{x}_1)}{\partial x_{01} \partial x_{02} \ldots \partial x_{0m}} \tag{126}$$

belong to a Boolean lattice defined by

$$\begin{aligned} & f_q^{\partial x_1 \partial x_2 \ldots \partial x_m}(\mathbf{x}_1) \wedge \overline{g_1(\mathbf{x})} \\ \vee & g_1(\mathbf{x}) \wedge f_r^{\partial x_1, \partial x_2, \ldots \partial x_m}(\mathbf{x}_1) \vee g_1^{id}(\mathbf{x}) = 0 \end{aligned} \tag{127}$$

with the mark functions of the m-fold derivatives with regard to $\mathbf{x}_0$

$$f_q^{\partial x_1 \partial x_2 \dots \partial x_m}(\mathbf{x}_1) = \frac{\partial^m f_q(\mathbf{x}_0,\mathbf{x}_1)}{\partial x_1 \partial x_2 \dots \partial x_m} \wedge \min_{\mathbf{x}_0}{}^m (f_q(\mathbf{x}) \vee f_r(\mathbf{x})) \ , \qquad (128)$$

$$f_r^{\partial x_1, \partial x_2, \dots \partial x_m}(\mathbf{x}_1) = \overline{\frac{\partial^m f_q(\mathbf{x}_0,\mathbf{x}_1)}{\partial x_1 \partial x_2 \dots \partial x_m}} \wedge \min_{\mathbf{x}_0}{}^m (f_q(\mathbf{x}) \vee f_r(\mathbf{x})) \ , \qquad (129)$$

and the independency function $g_1^{id}(\mathbf{x})$ associated to IDM(g_1) satisfies

$$\forall x_i \in \mathbf{x}_0 : \quad \text{MIDC}(\text{IDM}(g_1), x_i) = 0 \ ; \qquad (130)$$

all m-fold minimums

$$g_2(\mathbf{x}) = \min_{\mathbf{x}_0} f(\mathbf{x}_0, \mathbf{x}_1) \qquad (131)$$

belong to a Boolean lattice defined by

$$f_q^{\min^m_{\mathbf{x}_0}}(\mathbf{x}_1) \wedge \overline{g_2(\mathbf{x})} \vee g_2(\mathbf{x}) \wedge f_r^{\min^m_{\mathbf{x}_0}}(\mathbf{x}_1) \vee g_2^{id}(\mathbf{x}) = 0 \qquad (132)$$

with the mark functions of the m-fold minimums with regard to $\mathbf{x}_0$

$$f_q^{\min^m_{\mathbf{x}_0}}(\mathbf{x}_1) = \min_{\mathbf{x}_0}{}^m f_q(\mathbf{x}_0, \mathbf{x}_1) \ , \qquad (133)$$

$$f_r^{\min^m_{\mathbf{x}_0}}(\mathbf{x}_1) = \max_{\mathbf{x}_0}{}^m f_r(\mathbf{x}_0, \mathbf{x}_1) \ , \qquad (134)$$

and the independency function $g_2^{id}(\mathbf{x})$ associated to IDM(g_2) satisfies

$$\forall x_i \in \mathbf{x}_0 : \quad \text{MIDC}(\text{IDM}(g_2), x_i) = 0 \ ; \qquad (135)$$

all m-fold maximums

$$g_3(\mathbf{x}) = \max_{\mathbf{x}_0} f(\mathbf{x}_0, \mathbf{x}_1) \qquad (136)$$

belong to a Boolean lattice defined by

$$f_q^{\max^m_{\mathbf{x}_0}}(\mathbf{x}_1) \wedge \overline{g_3(\mathbf{x})} \vee g_3(\mathbf{x}) \wedge f_r^{\max^m_{\mathbf{x}_0}}(\mathbf{x}_1) \vee g_3^{id}(\mathbf{x}) = 0 \qquad (137)$$

with the mark functions of the m-fold maximums with regard to $\mathbf{x}_0$

$$f_q^{\max^m_{\mathbf{x}_0}}(\mathbf{x}_1) = \max_{\mathbf{x}_0}{}^m f_q(\mathbf{x}_0, \mathbf{x}_1) \ , \qquad (138)$$

$$f_r^{\max^m_{\mathbf{x}_0}}(\mathbf{x}_1) = \min_{\mathbf{x}_0}{}^m f_r(\mathbf{x}_0, \mathbf{x}_1) \ , \qquad (139)$$

and the independency function $g_3^{id}(\mathbf{x})$ associated to IDM(g_3) satisfies

$$\forall x_i \in \mathbf{x}_0 : \quad \text{MIDC}(\text{IDM}(g_3), x_i) = 0 \ ; \qquad (140)$$

and all Δ-operations

$$g_4(\mathbf{x}) = \Delta_{\mathbf{x}_0} f(\mathbf{x}_0, \mathbf{x}_1) \qquad (141)$$

belong to a Boolean lattice defined by

$$f_q^{\Delta_{\mathbf{x}_0}}(\mathbf{x}_1) \wedge \overline{g_4(\mathbf{x})} \vee g_4(\mathbf{x}) \wedge f_r^{\Delta_{\mathbf{x}_0}}(\mathbf{x}_1) \vee g_4^{id}(\mathbf{x}) = 0 \qquad (142)$$

with the mark functions of the Δ-operations with regard to $\mathbf{x}_0$

$$f_q^{\Delta_{\mathbf{x}_0}}(\mathbf{x}_1) = \max_{\mathbf{x}_0}{}^m f_q(\mathbf{x}_0, \mathbf{x}_1) \wedge \max_{\mathbf{x}_0}{}^m f_r(\mathbf{x}_0, \mathbf{x}_1) \ , \qquad (143)$$

$$f_r^{\Delta_{\mathbf{x}_0}}(\mathbf{x}_1) = \min_{\mathbf{x}_0}{}^m f_q(\mathbf{x}_0, \mathbf{x}_1) \vee \min_{\mathbf{x}_0}{}^m f_r(\mathbf{x}_0, \mathbf{x}_1) \ . \qquad (144)$$

and the independency function $g_4^{id}(\mathbf{x})$ associated to IDM(g_4) satisfies

$$\forall x_i \in \mathbf{x}_0 : \quad \text{MIDC}(\text{IDM}(g_4), x_i) = 0 \ . \qquad (145)$$

Proof: The proof for the first three m-fold derivative operations can be executed directly by an iterative execution of the steps to prove the sequence of simple derivatives. The lattice of Δ-operations with regard to $\mathbf{x}_0$ must be equal to 1 (143) when the subspace $\mathbf{x}_1 = const.$ contains both at least one function value 1 (138) and at least one function value 0 (134) and must be equal to 0 (144) when the subspace $\mathbf{x}_1 = const.$ contains either only function values 1 (133) or only function values 0 (139). Due to the restricted space we skip a more detailed proof. ∎

VI. Conclusion

This paper introduced a more general description of so far unused lattices of Boolean functions. It is shown that all derivative operations transform a given lattice into a simpler lattice if the given lattice depends on the used directions of change.

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