Functions Implicitly Defined by Logic Equations -
Unique Solutions

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Abstract

The problem of resolving restrictive logic equations occurs in the analysis and
synthesis of digital systems. The implicit representation of logic functions is useful to
describe the behavior of the systems, and the explicit unique representation is needed to
describe the structure. The main aim of this paper is to show an easy way to find
candidates of functions, resolving the given equation in a unique way, and to check that
these functions are valid. In order to do this, the basic definitions of derivative operations
are introduced, and, by using an inductive approach, suitable algorithms are suggested
and evaluated. Quantitative results about the uniquely resolvable equations show finally
some surprising results.

1 Introduction

In this paper we consider the restrictive logic equation
\[ f(x_1, \ldots, x_n, y) = 0 \] (1)
and look for a function
\[ y = \phi(x_1, \ldots, x_n) \] (2)
which is defined by this equation in a unique way. This means that the equation (1) should be resolved
in such a way that the unique function (2) satisfies the identity (3) for all patterns \( x \):
\[ f(x_1, \ldots, x_n, \phi(x_1, \ldots, x_n)) = 0 \] (3)
More generally, we have an restrictive logic equation
\[ f(x_1, \ldots, x_n, y_1, \ldots, y_m) = 0 \] (4)
and look for the set of functions
\[ y_1 = \phi_1(x_1, \ldots, x_n) \]
\[ \vdots \]
\[ y_m = \phi_m(x_1, \ldots, x_n) \] (5)
with
\[ f(x_1, \ldots, x_n, \phi_1(x_1, \ldots, x_n), \ldots, \phi_m(x_1, \ldots, x_n)) = 0 \] (6)
and all these functions (5) can be determined only in one way.
This problem goes back at least to Akers [1]. He found that the equation (4) is resolvable with
respect to \( \mathbf{y} = (y_1, \ldots, y_m) \) if and only if (7) is true.
A thorough investigation and a rather complete solution can be found in the monograph [2]. The most important results of [2] related to the topic of this paper are summarized in the following. A simpler solution (8) for the same problem (resolve equation (4) with respect to $y$) was stated and proved.

$$\min_{\mathcal{M}} m f(x_1, \ldots, x_n, y_1, \ldots, y_m) = 0$$  \hfill (8)

Based on the properties of the m-times minimum derivative shows [2] that the equation (4) is resolvable for any set $\mathcal{Y} = (y_1, \ldots, y_i, \ldots, y_m)$ including $y_i$ if (4) is resolvable with respect to $y_i$ itself. Next, in [2] the problem of the unique resolvability of the equation (4) is studied. Considering this property the searched functions can be calculated by (9).

$$\phi_i(x_1, \ldots, x_n, y_1, \ldots, y_m) = \frac{\partial^{m-1} f(x_1, \ldots, x_n, y_1, \ldots, y_m)}{\partial y_1 \partial y_{i-1} \partial y_{i+1} \ldots \partial y_m} \bigg|_{y_i=1}$$  \hfill (9)

Not each equation (4) is uniquely resolvable with respect to $\mathcal{Y} = (y_1, \ldots, y_m)$. In the case that (8) is valid, the functions (5) can be selected out of lattices specified in [2]. Finally, the conditions for unique resolvability are stated and proved in [2]; in the general case of equation (4) the criterion is (10)

$$\min_{\mathcal{M}} f(x_1, \ldots, x_n, y_1, \ldots, y_m) \bigg|_{y_i=0} = \min_{\mathcal{M}} f(x_1, \ldots, x_n, y_1, \ldots, y_m) \bigg|_{y_i=1}$$  \hfill (10)

and in the special case of equation (1) the criterion is (11).

$$\frac{\partial f(x_1, \ldots, x_n, y_i)}{\partial y_i} = 1$$  \hfill (11)

In [2] the property of uniqueness is related to the property of resolvability, but [4] shows that uniqueness and resolvability are independent properties in the context of logic equations. Both properties are studied in detail in [4] for the restrictive logic equations (1) and (4) and the dual characteristic logic equations as well, having the value 1 on the right side of the equation. In [4] several formulas are stated and proved to check the mentioned properties separately and in combination as well. For instance, [3] shows the criterion whether the equation (4) is uniquely resolvable with respect to $\mathcal{Y} = (y_1, \ldots, y_m)$ by (12).

$$\bigwedge_{i=1}^{k} \frac{\partial}{\partial y_i} \left[ \min_{\mathcal{M}} f(x_1, \ldots, x_n, y_1, \ldots, y_m) \right] = 1,$$

where $y_{0i} = (y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_m)$  \hfill (12)

However, the results presented in [4] are focused to the analysis of the properties and do not suggest directly how the searched functions can be synthesized (i.e. can be calculated) in the case, that a unique solution exists. The aim of this paper is to make both the analysis of the property of unique resolvability of restrictive logic equations and the synthesis of the solution functions more easily applicable.

The rest of the paper is organized as follows. Section 2 introduces selected derivatives of the Boolean Differential Calculus (BDC [2]) used in this paper. Section 3 shows an elegant and short way to find solution candidates of the restrictive logic equations (1) or (4) and offers an easy way of testing them. The quantitative analysis of uniquely resolvable restrictive logic equations is done in section 4. Finally, section 5 concludes the paper.
2 Preliminaries

Using the cofactors \( f(x_1,\ldots,x_i=0,\ldots,x_n) \) and \( f(x_1,\ldots,x_i=1,\ldots,x_n) \) of a Boolean function \( f(x_1,\ldots,x_n) \), the simple derivative operations are defined in (13), (14), and (15). Note: the results of these operations are Boolean functions too. All of them do not depend on the variable \( x_i \).

**Simple derivative:**
\[
\frac{\partial f(x_1,\ldots,x_i,\ldots,x_n)}{\partial x_i} = f(x_1,\ldots,x_i=0,\ldots,x_n) \oplus f(x_1,\ldots,x_i=1,\ldots,x_n) \tag{13}
\]

**Simple minimum:**
\[
\min_{x_i} f(x_1,\ldots,x_i,\ldots,x_n) = f(x_1,\ldots,x_i=0,\ldots,x_n) \land f(x_1,\ldots,x_i=1,\ldots,x_n) \tag{14}
\]

**Simple maximum:**
\[
\max_{x_i} f(x_1,\ldots,x_i,\ldots,x_n) = f(x_1,\ldots,x_i=0,\ldots,x_n) \lor f(x_1,\ldots,x_i=1,\ldots,x_n) \tag{15}
\]

The simple derivative is equal to one for such patterns \((x_1,\ldots,x_i-1,x_i+1,\ldots,x_n)\) where a change of the value of the variable \( x_i \) causes a change of the given function. The simple minimum is equal to one for such patterns \((x_1,\ldots,x_i-1,x_i+1,\ldots,x_n)\) where a change of the value of the variable \( x_i \) does not change the value one of the given function. Finally, the simple maximum is equal to one for such patterns \((x_1,\ldots,x_i-1,x_i+1,\ldots,x_n)\) where a change of the value of the variable \( x_i \) leads at least once to the value one of the given function.

The derivative operations introduced above can be executed iteratively for several variables. In this way \( m \)-times derivative operations are defined, see (16), (17), and (18).

**\( m \)-times derivative:**
\[
\frac{\partial^m f(x_{01},\ldots,x_{0i},\ldots,x_{lm})}{\partial x_{11}\partial x_{12}\ldots\partial x_{1m}} = \frac{\partial}{\partial x_{1m}} \left( \frac{\partial}{\partial x_{12}} \left( \frac{\partial}{\partial x_{11}} \left( \ldots \frac{\partial}{\partial x_{1l}} f(x_{01},\ldots,x_{li},\ldots,x_{lm}) \right) \right) \right) \tag{16}
\]

**\( m \)-times minimum:**
\[
\text{min}^m f(x_{01},\ldots,x_{0i},\ldots,x_{lm}) = \min_{x_{im}} \left( \ldots \min_{x_{i2}} \left( \min_{x_{i1}} f(x_{01},\ldots,x_{li},\ldots,x_{lm}) \right) \right) \tag{17}
\]

**\( m \)-times maximum:**
\[
\text{max}^m f(x_{01},\ldots,x_{0i},\ldots,x_{lm}) = \max_{x_{im}} \left( \ldots \max_{x_{i2}} \left( \max_{x_{i1}} f(x_{01},\ldots,x_{li},\ldots,x_{lm}) \right) \right) \tag{18}
\]

By means of the \( m \)-times minimum and the \( m \)-times maximum, the \( \Delta \)-operation (19), already used by Akers [1], can be defined.

**\( \Delta \)-operation:**
\[
\Delta f(x_{01},\ldots,x_{0i},\ldots,x_{lm}) = \text{min}^m f(x_{01},x_{i}) \oplus \text{max}^m f(x_{01},x_{i}) \tag{19}
\]

The results of all these \( m \)-times derivative operations depend only on the variables set \( x_0 \), and thus they are independent on the variables set \( x_1 \). The function values are defined by all function values of the subspaces \( x_1 = \text{constant} \). Table 1 shows the conditions for which a function value one in a \( m \)-times derivative operation occurs.

<table>
<thead>
<tr>
<th>( m )-times derivative operation</th>
<th>condition for a function value one in the result</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m )-times derivative</td>
<td>odd number of ones in the subspace ( x_1 = \text{constant} )</td>
</tr>
<tr>
<td>( m )-times minimum</td>
<td>only ones in the subspace ( x_1 = \text{constant} )</td>
</tr>
<tr>
<td>( m )-times maximum</td>
<td>at least one times a one in the subspace ( x_1 = \text{constant} )</td>
</tr>
<tr>
<td>( \Delta )-operation</td>
<td>not constant in the subspace ( x_1 = \text{constant} )</td>
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</tbody>
</table>
3 Resolve a Restrictive Logic Equation Uniquely

3.1 Single Function $\phi_1(x_1, \ldots, x_{n-1})$

We use an inductive approach and start with the equation (1). There are two questions. First, does there exist a function (2) such that (3) is true for each pattern of $x$. Second, how can the function (2) be found.

For later comparison we fix the number of variables of the function $f$ to the value $n$, labeling $n$-1 variables by $x_i, i = 1, \ldots, n$, and one variable by $y$. The task of this section is to resolve the equation

$$f(x_1, \ldots, x_{n-1}, y) = 0$$

in a unique way.

A very simple principle is used. Every vector $x = (\alpha_1, \ldots, \alpha_{n-1})$ can be extended by $y = 0$ and $y = 1$ which results in two vectors $(\alpha_1, \ldots, \alpha_{n-1}, 0)$ and $(\alpha_1, \ldots, \alpha_{n-1}, 1)$, respectively. There are exactly four functions $f$ for these two pattern labeled by $f_0$, $f_1$, $f_2$, and $f_3$ in table 2. Generally, each of this four functions may occur in a subspace $(\alpha_1, \ldots, \alpha_{n-1}) = \text{constant}$.

<table>
<thead>
<tr>
<th>Table 2. All possible subfunctions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$ ... $x_{n-1}$</td>
</tr>
<tr>
<td>$\alpha_1$ ... $\alpha_{n-1}$</td>
</tr>
<tr>
<td>$\alpha_1$ ... $\alpha_{n-1}$</td>
</tr>
</tbody>
</table>

The left-hand subfunction $f_0$ allows, that the $y$-value assigned to $(\alpha_1, \ldots, \alpha_{n-1})$ may be equal to 0 or 1. Thus, there does not exist a unique solution in case of $f_0$. The right-hand subfunction $f_3$ forbids both $y$-values and no function $y = \phi_1(x_1, \ldots, x_{n-1})$ can be found, which means that the solvability is lost altogether. Only the remaining subfunctions $f_1$ and $f_2$, highlighted in bold in table 2, enable a unique solution. The $f$-values for these two vectors have to be different (0 for the first, 1 for the second or vice versa). Since $y = \phi_1(x_1, \ldots, x_{n-1})$, we get

$$f(x_1, \ldots, x_{n-1}, y) = \phi_1(x_1, \ldots, x_{n-1}) \oplus y = 0.$$  \hspace{1cm} (21)

On the other side, based on the Shannon expansion, we have

$$f(x, y) = \overline{y} \cdot f(x, 0) \vee y \cdot f(x, 1) = \overline{y} \cdot f(x, 0) \oplus y \cdot f(x, 1)$$ \hspace{1cm} (22)

the following transformation can be used:

$$f(x, y) = (y \oplus 1) \cdot f(x, 0) \oplus y \cdot f(x, 1) \hspace{1cm} (23)$$

$$f(x, y) = f(x, 0) \oplus y \cdot (f(x, 0) \oplus f(x, 1))$$ \hspace{1cm} (24)

The comparison of the representations (21) and (24) in combination with the definition of the simple derivative (13) of results in the equations (25) and (26). The equation (25) describes an easy way, how the searched function $\phi_1(x_1, \ldots, x_{n-1})$ can be calculated. But, this synthesized function will be only valid if the analysis condition (26) is true for each pattern $(x_1, \ldots, x_{n-1})$.

$$\phi_1(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, 0)$$ \hspace{1cm} (25)

$$\frac{\partial f(x_1, \ldots, x_{n-1}, y)}{\partial y} = 1$$  \hspace{1cm} (26)
Hence, there are two very simple methods to resolve uniquely the equation (21), represented in Table 3.

**Table 3. Methods to resolve the equation (21) uniquely.**

<table>
<thead>
<tr>
<th>Method</th>
<th>Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>analysis followed by synthesis</td>
<td>Check the partial derivative (26) with regard to a (candidate) variable ( y ) whether it is identically equal to 1 and, if so, calculate ( \phi_1(x_1, ..., x_{n-1}) ) using (25).</td>
</tr>
<tr>
<td>synthesis followed by analysis</td>
<td>Calculate the only possible function ( \phi_1(x_1, ..., x_{n-1}) ) using (25) directly and check based on (21) whether the equation ( f(x_1, ..., x_{n-1}, y) = \phi_1(x_1, ..., x_{n-1}) \oplus y ) is valid for each pattern ((x_1, ..., x_{n-1}, y)).</td>
</tr>
</tbody>
</table>

If there are more than \( 2^{n-1} \) zeroes in the vector of \( f \) then the function \( \phi_1(x_1, ..., x_{n-1}) \) is no longer defined in a unique way, more than one function can be found for \( y \); if there are less than \( 2^{n-1} \) zeroes in the vector of \( f \) then the function for \( y \) cannot be defined anymore; for some vectors \((x_1, ..., x_{n-1})\) no zero can be found.

### 3.2 Set of Functions \([\phi_1(x_1, ..., x_{n-m}), \ldots, \phi_m(x_1, ..., x_{n-m})]\)

The transition to more than one variable \( y_1 \) follows the same idea. First, we consider the equation

\[
f(x_1, ..., x_{n-2}, y_1, y_2) = 0. \tag{27}
\]

In order to reach at a unique definition of \( y_1 \) and \( y_2 \), the value \( f = 0 \) must only appear once for the four possible combinations of \((y_1, y_2)\). Thus, only \( 2^2 = 4 \) functions of the \( 2^{2^2} = 16 \) possible functions in the subspace \((\alpha_1, ..., \alpha_{n-2}) = \text{constant}\) define a uniquely resolvable function \( f(x_1, ..., x_{n-2}, y_1, y_2) \). In table 4 these four functions are listed and labeled by \( u_i, i = 0, ..., 3 \).

**Table 4. Selected subfunctions, leading to a uniquely resolvable function \( f(x_1, ..., x_{n-2}, y_1, y_2) \)**

<table>
<thead>
<tr>
<th>( x_1 ) ... ( x_{n-2} ) ( y_1 ) ( y_2 )</th>
<th>( f_{00} )</th>
<th>( f_{01} )</th>
<th>( f_{u2} )</th>
<th>( f_{u3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha_1 ) ... ( \alpha_{n-2} ) 0 0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \alpha_1 ) ... ( \alpha_{n-2} ) 0 1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \alpha_1 ) ... ( \alpha_{n-2} ) 1 0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( \alpha_1 ) ... ( \alpha_{n-2} ) 1 1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Further 11 functions, depending on the two variables \( y_1 \) and \( y_2 \), include more than one zero and lead to not uniquely resolvable functions \( f(x_1, ..., x_{n-2}, y_1, y_2) \). The 16th remaining \( f(y_1, y_2) \) includes only ones and prevents the resolvability of (27). Only \( 2^{n-2} \) zeroes, which means 25 percent, are allowed in \( f(x_1, ..., x_{n-2}, y_1, y_2) \), and these zeroes must be distributed to the subspaces in such a way that exactly one zero occurs in each subspace defined by \((\alpha_1, ..., \alpha_{n-2}) = \text{constant}\).

The assumption is now that \( y_1 \) and \( y_2 \) are defined in a unique way:

\[
f(x_1, y_1, y_2) = \left[y_1 \oplus \phi_1(x_1, ..., x_{n-2})\right] \lor \left[y_2 \oplus \phi_2(x_1, ..., x_{n-2})\right] = 0, \tag{28}
\]

which results in the following equation (29).

\[
f(x_1, y_1, y_2) = \phi_1 \oplus \phi_2 \oplus \phi_1 \phi_2 \oplus y_1 \overline{\phi_1} \oplus y_2 \overline{\phi_1} \oplus y_1 y_2 = 0 \tag{29}
\]
On the other side, based on the Shannon expansion with respect of \( y_1, y_2 \):

\[
f(\bar{x}, y_1, y_2) = \bar{y}_1 y_2 f(\bar{x},0,0) \vee y_1 \bar{y}_2 f(\bar{x},1,0) \vee \bar{y}_1 y_2 f(\bar{x},0,1) \vee y_1 \bar{y}_2 f(\bar{x},1,1) = 0.
\]

Equation (30)

\[
f(x, y_1, y_2) = 0 \quad \text{can be represented by (31)}:
\]

\[
f(x, y_1, y_2) = f(x,0,0) \oplus y_1 f(x,1,0) \oplus y_2 f(x,0,1) \oplus y_1 y_2 f(x,1,1) = 0.
\]

Equation (31)

By comparing the representations (29) and (31), we receive the required solutions:

\[
\frac{\partial^2 f(x, y_1, y_2)}{\partial y_1 \partial y_2} = 1
\]

Equation (32)

\[
\frac{\partial f(x, y_1, y_2)}{\partial y_1} \bigg|_{y_2=0} = \phi_1(x)
\]

Equation (33)

\[
\frac{\partial f(x, y_1, y_2)}{\partial y_2} \bigg|_{y_1=0} = \phi_2(x)
\]

Equation (34)

\[
f(x,0,0) = \phi_1(x) \vee \phi_2(x)
\]

Equation (35)

The most important result will be again the possibility to determine \( \phi_1(x) \) and \( \phi_2(x) \) directly as the negated derivatives for given variables \( y_1, y_2 \). Using the above algorithm “synthesis followed by analysis” both functions must finally be checked based on (29). This check is obviously complete.

Alternatively the necessary check can be done using the formulas (32) and (35). It seems that this check may not be complete. (32) holds for all functions having an odd number of zeros in the subspace \((\alpha_1, \ldots, \alpha_{n-2}) = \text{constant}\). Thus, 50 percent of the possible functions are refused. Of cause, all of them are bad functions in the sense of the searched solution. The check by formula (35) evaluates only one pattern \((y_1 = 0, y_2 = 0)\) of the four possible pattern and extended the checked part by 25 percent.

What’s the matter with the remaining 25 percent? The answer to this question is connected to the property of the derivative, that the result of the derivation is the same for both a function and their complement. The remaining 25 percents of the necessary check are already included in the calculation by (33) and (34).

The algorithm “analysis followed by synthesis” from table 3 must be refined for the case of the functions \( \phi_1(x) \) and \( \phi_2(x) \). The first step is the analysis whether the 2-times derivative (32) holds. If that is not the case, the equation (27) is not uniquely resolvable with respect to \( (y_1, y_2) \). Only if there is the real possibility to find the functions \( \phi_1(x) \) and \( \phi_2(x) \), the only existing candidates must be synthesized by (33) and (34), respectively. The last step of this refined algorithm “analysis followed by synthesis and final check” is the check, whether (35) holds. Note that this final check is much easier than the complete check based on (29). The advantage of this algorithm is, that, if the necessary but not sufficient condition (32) does not hold, the algorithm finishes without further calculations.

The aim is to determine \( \phi_1(x) \) and \( \phi_2(x) \), but the formulas (33) and (34) describe, how their complement can be found. Known that the calculation of a complement is an expensive operation a simpler solution is welcome. The necessary condition (32) helps to find such a simpler solution. If (32) holds, the derivative (33) must be linear in the variable \( y_2 \) and the derivative (34) in the variable \( y_2 \), respectively. Based on this property we get the simpler formulas (36) and (37).

\[
\frac{\partial f(x, y_1, y_2)}{\partial y_1} \bigg|_{y_2=1} = \phi_1(x)
\]

Equation (36)

\[
\frac{\partial f(x, y_1, y_2)}{\partial y_2} \bigg|_{y_1=1} = \phi_2(x)
\]

Equation (37)
Note, these formulas correspond to the formula (9) known form [2].

In order to complete the inductive procedure, the case of three variables will be added. It is to be noticed that the number of zeroes of $f$ is reduced to 12.5 percent, only one value out of eight can be equal to 0. We consider the restrictive logic equation (38) and search the uniquely determined resolved functions (39).

$$f(x_1, \ldots, x_{n-3}, y_1, y_2, y_3) = 0$$  \hspace{1cm} (38)

$$y_1 = \phi_1(x_1, \ldots, x_{n-3}), \quad y_2 = \phi_2(x_1, \ldots, x_{n-3}), \quad y_3 = \phi_3(x_1, \ldots, x_{n-3})$$  \hspace{1cm} (39)

Repeating the method of comparing the coefficients of the expansion of $f(x_1, \ldots, x_{n-3}, y_1, y_2, y_3)$ and (40), we get formulas (41), … , (48).

$$f(x, y_1, y_2, y_3) = [y_1 \oplus \phi_1(x_1, \ldots, x_{n-3})] \lor [y_2 \oplus \phi_2(x_1, \ldots, x_{n-3})] \lor [y_3 \oplus \phi_3(x_1, \ldots, x_{n-3})] = 0$$  \hspace{1cm} (40)

$$\frac{\partial^3 f(x, y_1, y_2, y_3)}{\partial y_1 \partial y_2 \partial y_3} = 1$$  \hspace{1cm} (41)

$$\frac{\partial^2 f(x, y_1, y_2, y_3)}{\partial y_2 \partial y_3} \bigg|_{y_1=0} = \phi'(x)$$  \hspace{1cm} (42)

$$\frac{\partial^2 f(x, y_1, y_2, y_3)}{\partial y_1 \partial y_3} \bigg|_{y_2=0} = \phi'(x)$$  \hspace{1cm} (43)

$$\frac{\partial^2 f(x, y_1, y_2, y_3)}{\partial y_1 \partial y_2} \bigg|_{y_3=0} = \phi'(x)$$  \hspace{1cm} (44)

$$\frac{\partial f(x, y_1, y_2, y_3)}{\partial y_1} \bigg|_{y_2=0, y_3=0} = \phi'(x) \phi'(x)$$  \hspace{1cm} (45)

$$\frac{\partial f(x, y_1, y_2, y_3)}{\partial y_2} \bigg|_{y_1=0, y_3=0} = \phi'(x) \phi'(x)$$  \hspace{1cm} (46)

$$\frac{\partial f(x, y_1, y_2, y_3)}{\partial y_3} \bigg|_{y_1=0, y_2=0} = \phi'(x) \phi'(x)$$  \hspace{1cm} (47)

$$f(x, 0, 0, 0) = \phi'(x) \lor \phi'(x) \lor \phi'(x)$$  \hspace{1cm} (48)

The above discussed algorithm “analysis followed by synthesis and final check” is usable again. The analysis of the necessary condition (41) can be calculated quickly e.g. using the functions of [3], [5]. If (41) holds, the only possible unique solution functions are synthesized by (42), (43), and (44). Using these potential solution functions the final check is more complex and needs to evaluate the equations (45), (46), (47), and (48). Fortunately, all these four checks are only necessary if a unique solution of (38) exists. Otherwise the algorithm stops at the moment when one of the checks does not hold.

For more than three variables the procedure follows the same ideas, however, the number of zeroes will be smaller and smaller, and the solvability will be kept only in some more or less rare cases.
4 Number of Uniquely Resolvable Equation

In addition to the actual solution, the number of uniquely resolvable restrictive equations is interesting. We repeat the inductive approach.

First, the equation (21) is considered. Since there are $2^{n-1}$ different vectors $(\alpha_1, ..., \alpha_{n-1})$, the two $(2^1)$ possible combinations associated to the pattern of $y$ lead to

\[
n_{ur1} = \left(2^1\right)^{2^{n-1}}
\]

functions $f(x_1, ..., x_n, y)$ which define a function $y = \phi_i(x_1, ..., x_{n-1})$ in a unique way. The label $n_{ur1}$ means number of uniquely resolvable equations with respect to one variable $y$. The simplification $(2^1)^{2^{n-1}} = 2^{2^{n-1}}$ shows a reverse interpretation of this number of uniquely resolvable functions such that each one of the $2^{2^{n-1}}$ functions $y = \phi_i(x_1, ..., x_{n-1})$ creates exactly one of the uniquely resolvable functions $f(x_1, ..., x_{n-1}, y) = y \oplus \phi_i(x_1, ..., x_{n-1})$.

Second, the equation (27) is considered. Taking into account that there are $2^{n-2}$ different vectors $(\alpha_1, ..., \alpha_{n-2})$, and for each one of them one of the $2^2$ functions listed in table 4 can be chosen, the number of all uniquely resolvable equation (27) is defined by $n_{ur2}(28)$.

\[
n_{ur2} = \left(2^1\right)^{2^{n-2}}
\]

Third, by comparing (49) and (50), the number $n_{urm}$ of uniquely resolvable equations with respect to $m$ variables $y_i$ is given by (51), where $n$ is the total number of variables in the restrictive logic equation.

\[
n_{urm} = \left(2^m\right)^{2^{n-m}}, \quad m \leq n
\]

Table 5 evaluates (51) in the range $1 \leq m \leq n \leq 6$, and table 6 shows the percent values of (51) with regard to all $2^n$ functions. Note, for $n \geq 2$, there is no difference of the number of uniquely resolvable equations in both cases $m = 1$ and $m = 2$.

**Table 5. Number of uniquely resolvable restrictive logic equations.**

<table>
<thead>
<tr>
<th>$n$</th>
<th>$m$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>4</td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>3</td>
<td>16</td>
<td>16</td>
<td>8</td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>4</td>
<td>256</td>
<td>256</td>
<td>64</td>
<td>16</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>5</td>
<td>65536</td>
<td>65536</td>
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<td>256</td>
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</tr>
<tr>
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<td>4294967296</td>
<td>16777216</td>
<td>65536</td>
<td>1024</td>
<td>64</td>
<td></td>
</tr>
</tbody>
</table>

**Table 6. Percentage of uniquely resolvable restrictive logic equations.**

<table>
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<th>$n$</th>
<th>$m$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>50.00000%</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>25.00000%</td>
<td>25.00000%</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>6.25000%</td>
<td>6.25000%</td>
<td>3.12500%</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.39063%</td>
<td>0.39063%</td>
<td>0.09766%</td>
<td>0.02441%</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.00153%</td>
<td>0.00153%</td>
<td>0.00010%</td>
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<td></td>
</tr>
<tr>
<td>6</td>
<td>0.00000%</td>
<td>0.00000%</td>
<td>0.00000%</td>
<td>0.00000%</td>
<td>0.00000%</td>
<td>0.00000%</td>
<td></td>
</tr>
</tbody>
</table>
5 Conclusion

There are several algorithms to resolve a restrictive logic equation in an unique way. All of them combine analysis steps, which check whether certain conditions hold, and synthesis steps, in which the candidate functions are calculated. Based on the theory, these steps can be organized in several orders. The suggested best and general algorithm “analysis followed by synthesis and final check” limits the necessary calculation expenses in such a case where no unique solution exists. All steps of this algorithm are only necessary if there exists a unique solution for the selected variables.

The quantitative studies show like expected, that on the one hand the number of uniquely resolvable restrictive equations increase exponentially if the number of all variables increases linearly or the number of resolving variables is decreased. On the other hand, the share of uniquely resolvable restrictive equations decreases in such a way, that already in case of 6 variables less than $10^{-5}$ percent have a solution. A surprising result consists in the fact, that the number of the uniquely for two variables resolvable equations on $n$ variables is the same like that, taking only one variable to resolve the equation on $n$ variables in a unique way.

6 References