Solution of the Last Open Four-Colored Rectangle-free Grid - an Extremely Complex Multiple-Valued Problem

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***Abstract*—It is a challenge in the multi-valued domain to solve problems that depend on a large number of variables, as large as possible. We selected for this paper the problem of rectangle-free colorings using four colors which could not be solved so far for the grids of the sizes 12×21 and 21×12. This problem depends on $12 * 21 = 252$ four-valued variables. It is the last of so far unsolved rectangle-free grid problems for four colors.**

This paper aims at the solution of a multi-valued problem with an exceptionally high complexity. The search space for this finite problem is 4^{252} which is approximately $5.2 * 10^{151}$. A similar coloring problem was solved for the grid of the size 18×18 that relies on the extreme search space of approximately $1.1 * 10^{195}$. The construction of a cyclically reusable solution for a single color simplifies this search space approximately to $3.4 * 10^{97}$. Unfortunately, such a restriction to a single color is not possible in the case of a grid of the size 12×21. Hence, the complexity which must be handled in maintainable time grows additionally by a factor of more than 10^{54}.

Based on a very deep analysis of the properties of the problem we have constructed a strongly restricted SAT-model. This final model depends on 504 Boolean variables and 85.344 clauses. Using this SAT-instance we could calculate not only one solution but 38,926 representatives of different permutation classes of four-colored rectangle-free grids of the size 12×21.

***Keywords*-four-valued coloring, rectangle-free grid, Boolean equation, SAT-solver, Latin square.**

I. Introduction

There are many practical tasks which can be modeled and solved by graph coloring [4]. One of these tasks is the edge coloring of bipartite graphs without certain cycles. Bipartite graphs without cycles are used, for instance, for cryptographic mappings where cycles simplify attacks. We solve in this paper the last open problem for completely edge-colored bipartite graphs, which does not include cycles of the length four of edges colored by the same of four colors. We describe the bipartite graph by an adjacency matrix where rows represent source vertices and columns represent destination vertices. In the literature, an adjacency matrix is also called a grid.

A comprehensive theory for such grid colorings with regard to several fixed numbers of colors is published in [2]. In this paper, the problem is shortly defined as follows.

"A two-dimensional *grid* is a set $G_{m,n} = [m] \times [n]$. A grid $G_{m,n}$ is *c-colorable* if there is a function $\chi_{m,n} : G_{m,n} \rightarrow [c]$ such that there are no rectangles with all four corners the same color." In comparison with [2] we exchanged in this definition the variables m and n to get a natural alphabetic order of m rows and n columns.

Three properties specify the studied graph:

1) each source vertex is connected with each destination vertex,
2) each edge is colored by exactly one of four colors,
3) it is not allowed, that all edges of any quadruple of edges $\{e(s_1, d_1), e(s_1, d_2), e(s_2, d_1), e(s_2, d_2)\}$ are colored by the same color.

These three conditions mean in terms of the used data structure that all positions of the adjacency matrix must be colored by one of four colors, and it is not allowed that the four cross points of any pair of rows and columns use the same color.

Theorem 8.1 of [2] enumerates both lower bounds of grids, which are not rectangle-free four-colorable and upper bounds of grids which are rectangle-free four-colorable. The proof has been based on the Corollaries 2.4, 2.8 and 2.10 and the Theorems 4.6, 4.7, and 4.14 of the same paper. Theorem 8.1 of [2] includes the statements:

7) $G_{21,13}$ *and* $G_{13,21}$ *are not 4-colorable*, and
14) $G_{20,16}$ *and* $G_{16,20}$ *are 4-colorable*.

Hence, it cannot concluded whether $G_{12,21}$ and $G_{21,12}$ are rectangle-free four-colorable or not.

In [2] only 6 grids are mentioned for which it is so far unknown whether a rectangle-free four-coloring exists. These are the grids $G_{17,17}$, $G_{17,18}$, $G_{18,17}$, $G_{18,18}$, $G_{12,21}$, and $G_{21,12}$. In summer 2011 we found a method to calculate rectangle-free four-colored grids of the size $G_{17,17}$, $G_{17,18}$, $G_{18,17}$, and $G_{18,18}$. We published this method in [6] and [7]. As result of further research we found in February 2012 valid rectangle-free four-colored grids of the sizes $G_{12,21}$ and $G_{21,12}$ and put one of these solutions on the website of the blog [8]. In this paper, we describe the solution method for these last open rectangle-free four-colored grids.

The grid $G_{18,18}$ has $18 * 18 = 324$ positions, and one of four colors must be selected for each of them. Hence, there are in all $4^{324} = 1.16798 * 10^{195}$ different patterns in which one of four colors is assigned to each of the 18×18 positions of a grid. The grids $G_{12,21}$ and $G_{21,12}$ have $12 * 21 = 21 * 12 = 252$ positions so that *only* $4^{252} = 5.2374 * 10^{151}$ different patterns must be evaluated. Hence, it seems that it is easier to find rectangle-free four-colored grids of the size $G_{12,21}$ or $G_{21,12}$. However, the method published in [6] reduces the task to solve for the rectangle-free four-colored grid $G_{18,18}$ to a cyclic reusable pattern of a single color so that the search space could be reduced to $2^{324} = 3.4175 * 10^{97}$. Unfortunately, this simplification method cannot be reused for the grids $G_{12,21}$ and $G_{21,12}$ so that we have to bridge an additional gap of about $4^{252}/2^{324} = 1.5325 * 10^{54}$.

If there is a rectangle-free four-colored grid $G_{12,21}$ then the exchange of rows and columns of this grid $G_{12,21}$ results in a rectangle-free four-colored grid $G_{21,12}$. Therefore we restrict ourselves in the rest of the paper to the grid $G_{12,21}$. Assume that we are able to evaluate one of the $4^{252} = 5.2374 * 10^{151}$ patterns of the completely four-colored grids in one nano-second (10^{-9}) and spend 100 years ($3 * 10^9$). Then we must repeat the job $1.66 * 10^{133}$ times in order to know whether there is a correct color pattern and if YES which valid color pattern exists. Therefore, we see that we are going to solve an extremely complex problem.

The rest of the paper is organized as follows. In section 2 we repeat from [6] both a 4-valued model and a deduced Boolean model of the 4-color problem of grids because these basic models must be reused independent on the size of the grid. An analysis of the rectangle-free four-colored grid $G_{12,21}$ in Section III reveals valuable properties of the problem. The steps to solve the problem in Section IV require the construction of both a strongly entangled grid head and a well-restricted SAT-model. Before we conclude the paper we show the distribution of the 38,926 representatives of different permutation classes of four-colored rectangle-free grids $G_{21,12}$ to 4 Latin squares controlled by block and column permutations.

II. Model of 4-colorable Grids

A. 4-valued Model

The four colors can be represented by the four values $1, 2, 3, 4$. The value of the grid in the row r and the column c can be modeled by the 4-valued variable $x_{r,c}$. One rectangle of the grid is selected by the rows r_i and r_j and by the columns c_k and c_l. The color-condition for the rectangles can be described using the following three operations:

1) *equal (multi-valued):*

$$x \equiv y = \begin{cases} 1 & \text{if } x \text{ is equal to } y \\ 0 & \text{otherwise} \end{cases} , \qquad (1)$$

Table I
Mapping of the 4-valued color x to 2 Boolean variables a and b

x	a	b
1	0	0
2	1	0
3	0	1
4	1	1

2) *and (conjunction, binary):*

$$x \wedge y = \begin{cases} 1 & \text{if both } x \text{ and } y \text{ are equal to } 1 \\ 0 & \text{otherwise} \end{cases} , \qquad (2)$$

3) *or (disjunction, binary):*

$$x \vee y = \begin{cases} 0 & \text{if both } x \text{ and } y \text{ are equal to } 0 \\ 1 & \text{otherwise} \end{cases} . \qquad (3)$$

The function $f_{ec}(x_{r_i,c_k}, x_{r_i,c_l}, x_{r_j,c_k}, x_{r_j,c_l})$ (4) depends on four 4-valued variables and has a Boolean result that is true in the case that the colors in the corners of a rectangle selected by the rows r_i and r_j and by the columns c_k and c_l are identical.

$$\begin{aligned} & f_{ec}(x_{r_i,c_k}, x_{r_i,c_l}, x_{r_j,c_k}, x_{r_j,c_l}) = \\ & ((x_{r_i,c_k} \equiv 1) \wedge (x_{r_i,c_l} \equiv 1) \wedge (x_{r_j,c_k} \equiv 1) \wedge (x_{r_j,c_l} \equiv 1)) \vee \\ & ((x_{r_i,c_k} \equiv 2) \wedge (x_{r_i,c_l} \equiv 2) \wedge (x_{r_j,c_k} \equiv 2) \wedge (x_{r_j,c_l} \equiv 2)) \vee \\ & ((x_{r_i,c_k} \equiv 3) \wedge (x_{r_i,c_l} \equiv 3) \wedge (x_{r_j,c_k} \equiv 3) \wedge (x_{r_j,c_l} \equiv 3)) \vee \\ & ((x_{r_i,c_k} \equiv 4) \wedge (x_{r_i,c_l} \equiv 4) \wedge (x_{r_j,c_k} \equiv 4) \wedge (x_{r_j,c_l} \equiv 4)) \end{aligned} \qquad (4)$$

The condition that in the four corners of the rectangle selected by the rows r_i and r_j and by the columns c_k and c_l not only one of the four colors $1, 2, 3, 4$ appears is

$$f_{ec}(x_{r_i,c_k}, x_{r_i,c_l}, x_{r_j,c_k}, x_{r_j,c_l}) = 0 \ . \qquad (5)$$

For the whole grid $G_{m,n}$ we have the condition:

$$\bigvee_{i=1}^{m-1} \bigvee_{j=i+1}^{m} \bigvee_{k=1}^{n-1} \bigvee_{l=k+1}^{n} f_{ec}(x_{r_i,c_k}, x_{r_i,c_l}, x_{r_j,c_k}, x_{r_j,c_l}) = 0 \ . \qquad (6)$$

B. Binary Model

The next step is the mapping of the model into the Boolean space. The four color values can be expressed by two Boolean values. Table I shows the used mapping.

The mapping of the model into the Boolean domain requires a doubling of the number of variables but allows to skip the special comparison operation. The function (7) depends on eight Boolean variables and has a Boolean result that is true in the case that the colors in the corners of the rectangle selected by the rows r_i and r_j and by the columns c_k and c_l are identical.

$$
\begin{aligned}
&f_{ecb}(a_{r_i,c_k}, b_{r_i,c_k}, a_{r_i,c_l}, b_{r_i,c_l}, a_{r_j,c_k}, b_{r_j,c_k}, a_{r_j,c_l}, b_{r_j,c_l}) = \\
&\quad(\overline{a}_{r_i,c_k} \wedge \overline{b}_{r_i,c_k} \wedge \overline{a}_{r_i,c_l} \wedge \overline{b}_{r_i,c_l} \wedge \\
&\qquad \overline{a}_{r_j,c_k} \wedge \overline{b}_{r_j,c_k} \wedge \overline{a}_{r_j,c_l} \wedge \overline{b}_{r_j,c_l}) \vee \\
&\quad(a_{r_i,c_k} \wedge \overline{b}_{r_i,c_k} \wedge a_{r_i,c_l} \wedge \overline{b}_{r_i,c_l} \wedge \\
&\qquad a_{r_j,c_k} \wedge \overline{b}_{r_j,c_k} \wedge a_{r_j,c_l} \wedge \overline{b}_{r_j,c_l}) \vee \\
&\quad(\overline{a}_{r_i,c_k} \wedge b_{r_i,c_k} \wedge \overline{a}_{r_i,c_l} \wedge b_{r_i,c_l} \wedge \\
&\qquad \overline{a}_{r_j,c_k} \wedge b_{r_j,c_k} \wedge \overline{a}_{r_j,c_l} \wedge b_{r_j,c_l}) \vee \\
&\quad(a_{r_i,c_k} \wedge b_{r_i,c_k} \wedge a_{r_i,c_l} \wedge b_{r_i,c_l} \wedge \\
&\qquad a_{r_j,c_k} \wedge b_{r_j,c_k} \wedge a_{r_j,c_l} \wedge b_{r_j,c_l}) \qquad (7)
\end{aligned}
$$

The conditions of the 4-color problem on a grid $G_{m,n}$ are satisfied when the function f_{ecb} (7) is equal to 0 for all rectangles which can be expressed by

$$\bigvee_{i=1}^{m-1} \bigvee_{j=i+1}^{m} \bigvee_{k=1}^{n-1} \bigvee_{l=k+1}^{n} f_{ecb}(\mathbf{a}, \mathbf{b}) = 0 \ . \qquad (8)$$

III. Basic Consideration

The grid $G_{12,21}$ consists of 12 rows and 21 columns. Hence, $12 * 21 = 252$ elements must be colored with the given 4 colors. Due to the pigeonhole principle a rectangle-free assignment of at least $252/4 = 63$ grid elements with one selected color is a necessary condition for a rectangle-free four-colored grid $G_{12,21}$.

A first simple approach is the unique distribution of the 63 tokens of the same color to the 21 columns of the grid. For this assumption $63/21 = 3$ elements of each column of the grid $G_{12,21}$ have to be colored with the same selected color. At the first glance such an assignment fits well to the rectangle-free coloring of the grid $G_{12,21}$ with four colors. The remaining $12 - 3 = 9$ elements can be colored using each of the other three colors on three positions, too.

However, the evaluation of the color assignments in the row avoids this suggested simple approach. Dividing the necessary 63 elements by the 12 row results in 9 rows of 5 elements of the same color and 3 rows of 6 elements. It must be verified whether rectangle-free colorings of a grid $G_{12,6}$ for one color exist that hold the following conditions:

1) all 6 elements of one row of the grid $G_{12,6}$ are colored by the selected color,
2) each of the 6 columns of the grid $G_{12,6}$ contains 3 elements of the selected color,
3) the $3 * 6$ colored elements hold the rectangle-free condition (6) for $m = 12$ rows and $n = 6$ columns.

Theorem 1: There does not exist a rectangle-free coloring of the grid $G_{12,6}$ of 12 rows and 6 columns with $3*6 = 18$ elements colored by the same color such that one row is completely colored with the given color and each of the 6 columns contains 3 elements of the given color.

	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	0	0	0	0	0
3	1	0	0	0	0	0
4	0	1	0	0	0	0
5	0	1	0	0	0	0
6	0	0	1	0	0	0
7	0	0	1	0	0	0
8	0	0	0	1	0	0
9	0	0	0	1	0	0
10	0	0	0	0	1	0
11	0	0	0	0	1	0
12	0	0	0	0	0	1

Figure 1. Assignment of the color 1 to the grid $G_{12,6}$ based on the proof of Theorem 1

Proof: The exchange of any pair of rows or any pair of columns within each grid does not change the property that the grid holds the rectangle-free condition. Hence, without loss of generality the first row can be chosen to color all included elements with the same given color. For the same reason the required further two elements with the same color in the first column can be located in the rows number 2 and 3. Figure 1 shows these assignments.

In order to hold the rectangle-free condition no further element of the rows 2 and 3 can be colored with the given color. As shown in Figure 1, the required assignments of three colored elements in the columns 2, 3, 4, and 5 fill up the rows from number 4 to number 11. The rectangle-free condition prohibits each further assignment of the given color to an element of the rows from 2 to 11 to the grid shown in Figure 1 so that the column 6 contains only two elements colored rectangle-free by the given color. Each further assignment of the given color to any element of the column 6 in the grid shown in Figure 1 violates the rectangle-free condition. ■

The conclusion of Theorem 1 is that there does not exist a rectangle-free four-coloring of the grid $G_{12,21}$ which contains in each column each of the four colors three times. The restriction to three or two elements of the same color in the columns of the grid $G_{12,21}$ violates, due to the pigeonhole principle, the requirement of a complete coloring of the grid. Taking into account that each of the three necessary rows of six elements of the same color causes a column of only two elements of this color, three columns of four elements colored with this color must be used.

There are several possibilities to distribute four elements of the same color in the three necessary columns. Due to the rectangle-free condition, it is not correct that these assignments overlap in more than one row. The three columns of four elements colored by the same color cause the smallest restriction for the remaining columns when the color assignments do not overlap in any row.

	1	2	3
1	1	0	0
2	1	0	0
3	1	0	0
4	1	0	0
5	0	1	0
6	0	1	0
7	0	1	0
8	0	1	0
9	0	0	1
10	0	0	1
11	0	0	1
12	0	0	1

Figure 2. Grid $G_{12,3}$ of disjoint assignments of the color 1 to four rows in three columns

	1			4				8				12				16			
1	1	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
2	1	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
3	1	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0
4	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
5	0	1	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0
6	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0
7	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0
8	0	1	0	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1
9	0	0	1	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1
10	0	0	1	0	1	0	0	1	0	0	0	0	0	0	1	0	0	1	0
11	0	0	1	0	0	0	1	0	0	1	0	1	0	0	0	0	1	0	0
12	0	0	1	0	0	1	0	0	0	0	1	0	1	0	0	1	0	0	0

Figure 3. Assignment of the color 1 to three elements in the columns from number 4 to number 19 to the grid $G_{12,19}$ based on the proof of Theorem 2

Despite of the maximal freedom for further assignments of the same color the chosen distribution of color assignments of Figure 2 restricts the maximal number of columns in which three elements can be colored with the same color.

Theorem 2: The grid $G_{12,3}$ of Figure 2 can be extended to a rectangle-free grid of maximal 16 columns which include three elements of the same color. Without violating the rectangle-free condition additional columns can contain only in a single row the same color.

Proof: Figure 3 shows a correct rectangle-free grid $G_{12,19}$ that holds the conditions of Theorem 2. Due to the four colored elements in column 1 the rectangle-free condition restricts the color assignments in the rows from 1 to 4 to a single grid element in the columns from 4 to 19. Analog restrictions must hold for the interval of rows from 5 to 8 caused by the second column and for the interval of rows from 9 to 12 caused by the third column, respectively. Hence, the three assignments of the given color in the columns from 4 to 19 must be done in the three row intervals from 1 to 4, from 5 to 8, and from 9 to 12.

The rectangle-free condition bans the assignment of the given color in two columns to single selected rows of the three row intervals. Hence, the assignment of the given color in the first row is restricted to four columns due to the four rows in the second and third row interval. The well-ordered pattern of color assignment in the interval of the rows from 1 to 4 in Figure 3 can be reached be permutations of columns 4 to 19. The used color assignment in Figure 3 in the interval of rows from 5 to 8 in a downstairs order is one simple possible selection. The rule to be satisfied is that each assignment in the interval of the rows from 1 to 4 is combined with one assignment in the interval of the rows from 5 to 8. These $4 * 4 = 16$ combinations restrict the maximal number of columns with three color assignments under the restriction of the assignments in the first three columns to 16 additional columns. This proves the first assertion.

The assignments in the last interval of rows from 9 to 12 must cover each row and each column in the intervals from 4 to 7, from 8 to 11, from 12 to 15, and from 16 to 19 exactly once. There are several possible assignments in this range, which hold the rectangle-free condition.

An exhaustive evaluation of the color assignments in Figure 3 proves that each pair of rows is covered by the assignment of the given color. Other possible assignments do not change this property. This proves the second assertion that without violating, the rectangle-free condition additional columns can contain the same color only in a single row. ■

Using permutations of rows and columns the pattern in the upper 8 rows and the left 3 columns of Figure 3 can be constructed. The rectangle-free condition in the remaining region can be reached by means of Latin squares [3]. Figure 4 shows the four reduced Latin squares of the size 4×4, which have a fixed natural order of the letters in the first row and the first column. These reduced Latin squares are labeled by the values 0, ..., 3 based on the letters of the main diagonal in natural order. The mapping rules between a Latin square and the four blocks, which are indicated by thick lines in the bottom right of Figure 3 are as follows:

- one row of the last four blocks is mapped to one row of the Latin square,
- the letter a is mapped to the value 1 in the leftmost position of a block,
- the letter b is mapped to the value 1 in the second position of a block from the left,
- the letter c is mapped to the value 1 in the third position of a block from the left, and
- the letter d is mapped to the value 1 in the rightmost position of a block.

All correct patterns of the selected color can be constructed based on the four Latin squares of Figure 4 and permutations restricted to the last four rows and complete 4×4 blocks in these rows. Figure 3 shows the result of the mapping of

0

a	b	c	d
b	a	d	c
c	d	a	b
d	c	b	a

1

a	b	c	d
b	a	d	c
c	d	b	a
d	c	a	b

2

a	b	c	d
b	c	d	a
c	d	a	b
d	a	b	c

3

a	b	c	d
b	d	a	c
c	a	d	b
d	c	b	a

Figure 4. All four reduced Latin squares $0, \dots, 3$ of the size 4×4

1	0	0	1	1	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
0	1	0	1	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0
0	1	0	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0
0	1	0	0	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0
0	1	0	0	0	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1
0	0	1	0	1	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1
0	0	1	0	0	0	1	0	0	1	0	0	0	0	0	0	1	0	0	1	0
0	0	1	0	0	0	0	0	1	0	0	1	0	1	0	0	0	0	1	0	0
0	0	1	0	0	0	0	1	0	0	0	0	1	0	1	0	0	1	0	0	0

Figure 5. Rectangle-free grid $G_{12,21}$ that contains 63 tokens of the same color

Latin square 1 without permutations of rows or blocks.

Now we can check whether the color assignment of Figure 3 can be extended to the necessary 63 color assignments in a grid $G_{12,21}$. The grid $G_{12,19}$ of Figure 3 contains three columns of four color tokens and 16 columns of three color tokens which are $3*4+16*3 = 12+48 = 60$ color tokens. The additional two columns 20 and 21 can include only a single color token due to Theorem 2. Hence, such a grid $G_{12,21}$ covers only $60 + 2 = 62$ color tokens and does not reach the required limit of 63 color tokens for a possible completely four-colored rectangle-free grid $G_{12,21}$.

Does this observation prove that no four-colored rectangle-free grid $G_{12,21}$ exists? Our answer is: NO. One of the columns 4 to 19 of Figure 3 can be replaced by three columns, which include two color tokens each. Using column 4 for this split we get the required

$$3 * 4 + 3 * 2 + 15 * 3 = 12 + 6 + 45 = 63$$

color assignments to construct a completely four-colored rectangle-free grid $G_{12,21}$. Figure 5 shows such a grid $G_{12,21}$ that contains a rectangle-free assignment of 63 tokens of the same color 1.

IV. Steps to Find the Four-colored Rectangle-free Grid $G_{12,21}$

A. Merge of Grid Heads of all Colors

We call the 6 columns of four or two tokens of the same color the *head of the grid*. The remaining 15 columns of 3 tokens of the same color are called *body of the grid*.

Each column of the grid must be completely filled by the four colors. This can be achieved by merging four tokens

1	1	2	2	1	1	2
2	1	0	0	0	0	2
3	1	0	0	0	0	2
4	1	0	0	0	0	2
5	2	1	2	1	2	1
6	0	1	0	0	2	0
7	0	1	0	0	2	0
8	0	1	0	0	2	0
9	2	2	1	2	1	1
10	0	0	1	2	0	0
11	0	0	1	2	0	0
12	0	0	1	2	0	0

Figure 6. Rectangle-free grid $G_{12,6}$ that merges the heads of two colors

2	1	1	2	2	3	3
2	3	4	1	4	2	4

1	1	0	0	0	0
0	0	2	2	0	0
0	0	0	0	3	3
0	4	0	4	0	0
1	0	1	0	0	0
0	0	2	0	2	0
3	0	0	0	3	0
0	4	0	0	0	4
0	1	1	0	0	0
0	0	0	2	2	0
3	0	0	0	0	3
0	0	0	4	0	4

(a)

4	2	2	3	3	1	1
4	4	3	4	1	4	2

1	1	3	**1**	4	2
4	3	2	2	1	**2**
2	**3**	4	1	3	3
4	4	3	4	1	2
1	2	1	3	4	**1**
4	**2**	2	3	2	1
3	2	4	**3**	3	1
2	4	3	1	**4**	4
4	1	1	3	**1**	2
2	3	4	2	2	1
3	2	**3**	1	4	3
2	3	**4**	4	1	4

(b)

4	2	2	3	3	1	1
4	4	3	4	1	4	2

1	1	4	3	**1**	2
2	3	2	2	4	1
4	2	**3**	1	3	3
2	4	3	4	**4**	1
1	3	1	**1**	4	2
4	3	2	1	2	**2**
3	**3**	4	1	3	2
2	4	**4**	3	1	4
2	1	1	3	4	**1**
4	**2**	3	2	2	1
3	2	4	**3**	1	3
4	2	3	4	1	4

(c)

Figure 7. Rectangle-free grid $G_{12,6}$ that merges the heads of all four colors: (a) assignment of two-token columns, (b) first extension of (a) with four-token columns, (c) second extension of (a) with four-token columns

of the first color, two tokens of the second color and three tokens of the remaining two colors each. Figure 6 shows the grid head that contains the four- and two-token columns of the first two colors within six columns.

The grid head shown in Figure 6 cannot be extended to a four-colored rectangle-free grid $G_{12,21}$. This conclusion follows form Theorem 1 and necessary extensions of the rows 1, 5, and 9 by 3 tokens of the colors 1 and 2, and 5 tokens of the colors 3 and 4. The sum of these tokens is 22 but there are only 21 columns.

The construction of the grid head of six columns of all four colors is the key to our solution of the four-colored rectangle-free grid $G_{12,21}$. This construction requires complicated entanglements of the assignments of two- or four-token-columns of the four colors. We explain our sequence of decisions supported by Figure 7.

From the trivial merging of the grid heads of two colors in Figure 6 we learned that the tokens of the two-token columns of different colors must be assigned to different rows. We decided further that the triples of two-token columns overlap in one column. The array on top of Figure 6 (a) indicates the columns, which are used for two tokens of the specified color. It can be seen that the first three colors can be assigned in continues ranges in a cyclic manner. Consequently, the columns 2, 4, and 6 must be used as two-token columns of color number 4.

It is known from the construction of the single-color grid head that the four-token columns of one color do not overlap with the two-token columns of the same color. Based on the selected two-token columns of Figure 7 (a) the four-token columns of the grid head must be chosen as shown in the array on top of Figure 7 (b) and (c).

As shown in Figure 2, each row must contain exactly one token in the four-token columns. Figure 7 (a) shows that up to now each row leaves four elements in the grid head free. Hence, each of the four colors must be assigned exactly once to these four free elements labeled by 0 in the rows of Figure 7 (a). It can be seen from the array on top of Figure 7 (b) and (c) that this rule can be satisfied in each row by one of the four colors only in a single position. These necessary positions are labeled by thick boarders in Figure 7 (b) and (c).

We know from Figure 5 that each two-token column overlaps in two rows of the grid head with two four-token columns of the same color. There are two possible positions for these assignments in each row used for the two-token columns. The assignment of one colored token to one of these grid elements entails two further assignments of the same color in other rows and columns. Bold numbers in Figure 7 (b) and (c) show the chosen assignments of the color number i in rows of number j with $j \mod 4 = i$.

Due to the rule that the free elements in Figure 7 (a) must contain each of the four colors in one element the colors of the remaining two free elements in each row are implicitly defined. Figure 7 (b) shows a four-colored rectangle-free grid $G_{12,6}$ in which the heads of all four colors are merged.

Based on the two options of the bold numbers in Figure 7 (b) for each of the four colors there are altogether $2^4 = 16$ different assignments. A detailed analysis reveals that only two of these assignments including the necessary subsequent assignments satisfy the condition of the four-token columns. Figure 7 (c) shows the second valid four-colored rectangle-free grid $G_{12,6}$ in which the heads of all four colors are merged.

B. Extended Models to Solve the Grid Using a SAT-solver

It is the aim of the satisfiability problem (shortly SAT) to find at least one assignment of Boolean variables such that a Boolean expression in conjunctive form becomes true. The power of SAT-solvers [1] has improved over the last decades forced by several SAT-competitions. As in [6] and [7] we tried to find a four-colored rectangle-free grid $G_{12,21}$ using the best SAT-solvers from the SAT-competitions of the last years. Equation (8) can easily be transformed into a SAT-equation by negation of both sides and the application of de Morgan's law to the Boolean expression on the lefthand side. In this way we get the required conjunctive form for the SAT-solver (9):

$$\bigwedge_{i=1}^{m-1} \bigwedge_{j=i+1}^{m} \bigwedge_{k=1}^{n-1} \bigwedge_{l=k+1}^{n} \overline{f_{ecb}(a_{r_i,c_k}, b_{r_i,c_k},} \\ \overline{a_{r_i,c_l}, b_{r_i,c_l}, a_{r_j,c_k}, b_{r_j,c_k}, a_{r_j,c_l}, b_{r_j,c_l})} = 1 \quad . \qquad (9)$$

The SAT-formula (9) for the grid $G_{12,21}$ depends on $12 * 21 * 2 = 504$ Boolean variables and contains $\binom{12}{2} * \binom{21}{2} * 4 = 55,440$ clauses. Based on comparable results of [6] it can be expected that the SAT-solver needs more than $2 * 10^{15}$ years for this solution.

Using the grid head of Figure 7 (b) or (c) $12 * 6 * 2 = 144$ Boolean variable can be used with fixed values. This simplifies the problem by a factor of $2^{144} \approx 2.23 * 10^{43}$. Due to this strong simplification it could be expected that the used SAT-solver `clasp-2.0.0-st-win32` finds the solution within few seconds. However, the additional restrictions of the grid head exclude so many color patterns that after a runtime of one month no solution was found.

In the next experiment we have extended the SAT-equation (9) using our knowledge of the structure of a possible solution. We know that each of the 15 columns from number 7 to number 21 must contain each of the four colors exactly 3 times. Hence, it is incorrect that the same color occurs four times in these columns. We generated a SAT-formula that contains the 55,440 clauses for the rectangle rule (9), the 144 clauses of the constant values of the grid head and additionally $\binom{12}{4} * 15 * 4 = 29,700$ clauses for the color restriction in the body of the grid $G_{12,21}$. Unfortunately, the SAT-solver did not find a solution for this more precise SAT-formula of 504 Boolean variables and 85,284 clauses within one month.

Due to the very long runtime of the SAT-solver, we restricted the search space of the problem further in the next experiment. We mapped the potential solution of the body of the grid $G_{12,21}$ for the first color as shown in Figure 5 to the entangled position based on both grid heads of Figure 7 (b) and (c). The SAT formula of 504 Boolean variables contains with this extension 85,374 clauses, but the free variables are restricted to $15 * 9 * 2 = 270$. The SAT-solver solves this restricted problem for the grid head of Figure 7 (b) within only 0.608 seconds and for the grid head of Figure 7 (c) within only 0.069 seconds, respectively. However, the answer of the SAT-solver was in both cases `UNSATISFIABLE`.

Does this result mean that no rectangle-free four-colored grid $G_{12,21}$ exists? Our answer to this question is *NO*. We explored both unsatisfiable SAT problems more in detail. The embedded grid heads of Figure 7 (b) and (c) can be

1	1	3	1	4	2	1	1	1	3	4	3	2	4	2	3	2	4	3	4	2
4	3	2	2	1	2	1	3	3	2	2	1	3	4	4	4	1	1	3	2	4
2	3	4	1	3	3	2	4	4	1	1	1	1	4	3	3	3	2	4	2	2
4	4	3	4	1	2	4	2	1	1	4	2	3	3	1	2	3	2	4	1	3
1	2	1	3	4	1	4	4	3	1	3	2	4	1	4	3	2	1	2	2	3
4	2	2	3	2	1	1	3	4	3	1	4	4	2	1	2	3	4	1	3	2
3	2	4	3	3	1	3	1	2	2	4	1	4	3	2	1	4	2	3	1	4
2	4	3	1	4	4	3	3	4	2	3	2	2	2	3	4	4	1	1	1	1
4	1	1	3	1	2	2	4	2	4	1	3	3	2	3	1	4	3	2	4	1
2	3	4	2	2	1	4	2	1	3	3	4	1	3	2	4	1	3	2	4	1
3	2	3	1	4	3	2	2	3	4	2	4	2	1	1	1	1	3	4	3	4
2	3	4	4	1	4	3	1	2	4	2	3	1	1	4	2	2	4	1	3	3

1	1	4	3	1	2	2	2	3	1	4	4	3	1	2	4	3	1	3	4	2
2	3	2	2	4	1	4	2	1	4	2	1	4	3	1	3	3	1	2	4	3
4	2	3	1	3	3	2	4	2	4	2	4	3	2	3	3	4	1	1	1	1
2	4	3	4	4	1	1	3	3	1	3	4	2	2	2	2	1	4	4	1	3
1	3	1	1	4	2	1	1	1	2	3	2	3	2	4	4	4	3	2	3	4
4	3	2	1	2	2	3	3	2	1	1	1	1	4	3	2	2	4	3	4	4
3	3	4	1	3	2	4	4	3	3	2	3	2	1	1	1	1	2	4	2	4
2	4	4	3	1	4	1	3	2	2	1	3	4	4	1	3	4	2	1	3	2
2	1	1	3	4	1	2	4	4	3	4	2	1	4	3	1	2	3	1	2	3
4	2	3	2	2	1	3	1	4	3	1	2	4	1	2	4	3	2	4	3	1
3	2	4	3	1	3	3	1	4	4	3	1	2	3	4	1	2	4	2	1	2
4	2	3	4	1	4	4	2	1	2	4	3	1	3	4	2	1	3	3	2	1

Figure 8. Rectangle-free four-colored grids $G_{12,21}$; the left grid extends the grid head of Figure 7 (b) and the right gird extends the grid head of Figure 7 (c)

transformed by permutation of rows and columns in such a way that the patterns of the color number 1 is equal to the pattern of this color shown in the first six columns of Figure 5. The pattern of the color number 1 in the remaining columns from 7 to 21 and the rows from 1 to 8 can be transformed into the pattern of this range given in Figure 5 using only permutations of columns. Hence, it remains a region of several choices of the color number 1 in the region of columns from 7 to 21 and rows from 9 to 12. We removed the 15 assignments of the color number 1 in this region and get a SAT formula of 504 Boolean variables and 85344 clauses that includes 300 free variables. It took 343.305 seconds to solve this problem for the grid head of Figure 7 (b) and 577.235 seconds for the grid head of Figure 7 (c) using the SAT-solver `clasp-2.0.0-st-win32`. Figure 8 shows these solutions of four-colored rectangle-free grids $G_{12,21}$ for both grid heads of Figure 7.

V. Classes of Four-Colored Rectangle-free Grids $G_{12,21}$

Many solution grids can be constructed by permutations of rows and / or columns based on the four-colored rectangle-free grids $G_{12,21}$ of Figure 8 which have been found first. We call such a solution set an equivalence class. It is an interesting question, how many different classes of four-colored rectangle-free grids $G_{12,21}$ exist for the grid heads of Figure 7 which cannot be found by such permutations.

This problem can be solved in such a way that all solutions of the finally constructed SAT-formula of 504 variables and 85,344 clauses are calculated and evaluated regarding their equivalence class. The complete calculation takes only 453.880 seconds the grid head of Figure 7 (b) and 745.700 seconds for the grid head of Figure 7 (c). There are 38,926 different solutions for each of these grid heads. Regarding the different fixed patterns of the color number 1 these solutions are divided into 44 classes. These classes can be mapped to the four reduced Latin squares of the size 4×4 by permutations of the last three lines and the last three blocks of four columns. The restrictions for permutations of three elements result from the fixed top left element of the leftmost block in the grid head.

Figure 9 shows the distribution of the found 44 classes regarding the possible permutations of the four Latin squares for both grid head. This analysis has been based on the constant assignment of the color 1 in the range of the rows form 1 to 8 and the columns from 7 to 21 as shown in Figure 5. All possible remaining assignments of color 1 in the range of the rows form 9 to 12 and the columns from 7 to 21 ordered by the reached reduced Latin squares (a), ..., (d), as result of the given permutations of the last 3 blocks of four columns and the last three rows.

The values in the eight tables of Figure 8 indicate the number of different assignments of the colors number 2, 3, and 4 for a certain fixed assignment of the first color. A value 0 in these tables means that no four-colored rectangle-free grid $G_{12,21}$ exists for the chosen assignment of the color number 1. The unsatisfiable SAT-formulas which, extend the gird heads of Figure 7 (b) and (c) by all assignments of the color number 1 in the columns from 7 to 21 as shown in Figure 5 belong to the class of the top left corner in Table (b1) or (a2) of Figure 9.

VI. Conclusion

We explored in this paper the so far unsolved problem whether grids $G_{12,21}$ and $G_{21,12}$ are rectangle-free four-colorable. We found $4.6 * 10^{34}$ rectangle-free four-colored grids $G_{12,21}$ out of the unimaginably large number of $5.2374 * 10^{151}$ possible assignments of 4 colors to the grid.

The way to find rectangle-free four-colorable grids $G_{12,21}$ is in detail completely different from the way to solve the equivalent problem for the grids $G_{18,18}$. It is not possible to solve the grids $G_{12,21}$ by restriction to a single color. Knowing the solutions of both problems we can conclude that the $G_{12,21}$ is much more complicated than the grid $G_{18,18}$. However, we can confirm that the general approach to solve such an extremely complex problem consists in a evaluation for the grid head of Figure 7 (b)

(a1) numbers of permutation classes for the Latin square 0

permutation of rows	permutation of blocks					
	123	132	213	312	231	321
123	875	0	0	0	0	16
132	0	16	0	70	0	0
213	15720	0	0	0	0	875
312	0	0	0	0	0	0
231	0	875	0	16	0	0
321	0	0	0	0	0	0

(b1) numbers of permutation classes for the Latin square 1

permutation of rows	permutation of blocks					
	123	132	213	312	231	321
123	0	184	0	16	0	0
132	10	0	0	0	0	48
213	0	5488	0	185	0	0
312	0	0	0	0	0	0
231	822	0	0	0	0	10
321	0	0	0	0	0	0

(c1) numbers of permutation classes for the Latin square 2

permutation of rows	permutation of blocks					
	123	132	213	312	231	321
123	185	0	2	0	0	10
132	0	185	0	16	0	0
213	5488	0	0	0	0	822
312	0	2	32	0	20	0
231	0	10	0	48	0	0
321	0	0	0	20	0	0

(d1) numbers of permutation classes for the Latin square 3

permutation of rows	permutation of blocks					
	123	132	213	312	231	321
123	5488	0	0	0	0	184
132	0	16	20	48	0	0
213	822	0	0	0	0	10
312	0	0	2	0	0	32
231	0	184	32	10	4	0
321	0	0	0	0	4	4

evaluation for the grid head of Figure 7 (c)

(a2) numbers of permutation classes for the Latin square 0

permutation of rows	permutation of blocks					
	123	132	213	312	231	321
123	0	70	0	16	0	0
132	16	0	0	0	0	875
213	0	0	0	0	0	0
312	875	0	0	0	0	15720
231	0	0	0	0	0	0
321	0	16	0	875	0	0

(b2) numbers of permutation classes for the Latin square 1

permutation of rows	permutation of blocks					
	123	132	213	312	231	321
123	48	0	0	0	0	10
132	0	16	0	184	0	0
213	0	0	0	0	0	0
312	0	185	0	5488	0	0
231	0	0	0	0	0	0
321	10	0	0	0	0	822

(c2) numbers of permutation classes for the Latin square 2

permutation of rows	permutation of blocks					
	123	132	213	312	231	321
123	0	16	0	185	0	0
132	10	0	0	0	2	185
213	0	0	20	2	32	0
312	822	0	0	0	0	5488
231	0	20	0	0	0	0
321	0	48	0	10	0	0

(d2) numbers of permutation classes for the Latin square 3

permutation of rows	permutation of blocks					
	123	132	213	312	231	321
123	0	48	0	16	20	0
132	184	0	0	0	0	5488
213	32	0	0	0	2	0
312	10	0	0	0	0	822
231	4	0	4	0	0	0
321	0	10	4	184	32	0

Figure 9. Number of permutation classes of four-colored rectangle-free grids $G_{12,21}$

deep analysis and the creation of models which cover not only the basic requirements but utilize also their hidden properties.

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